Topics in Macroeconomics

Unit 3 – Introduction

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February 2023

Outline

Outline of unit 3

1. Introduction (this mini-lecture)
   - Recursive methods in quantitative macroeconomics
   - Infinite-horizon vs. life-cycle solution methods
2. Value function iteration (VFI)
3. Endogenous grid-point method (EGM)

Recursive formulation of household problem

Recall that we can write a household problem in two ways:

1. Sequential formulation

$$
\begin{aligned}
V(a_0, y_0) = \max_{\{c_t\}_{t=0}^{\infty}, \{a_{t+1}\}_{t=0}^{\infty}} & \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t) \middle| y_0 \right] \\
\text{s.t.} \quad & c_t + a_{t+1} = (1+r)a_t + y_t \quad \forall\, t \\
& c_t \geq 0, \;\; a_{t+1} \geq 0 \quad \forall\, t
\end{aligned}
$$

2. Recursive formulation

$$
\begin{aligned}
V(a, y) = \max_{c,\, a'} & \left\{ u(c) + \beta \mathbb{E}\left[V(a', y') \middle| y \right] \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a + y \\
& c \geq 0, \;\; a' \geq 0
\end{aligned}
$$

Recursive methods

- The sequential formulation is quite useless for solving heterogeneous-agent models numerically

  $\Rightarrow$ We exclusively deal with recursive formulation
- We want to find functions that characterise the solution:
  1. The value function $V(a, y)$
  2. The policy functions

$$\begin{aligned} c &= C(a, y) \quad \text{Optimal consumption} \\ a' &= A(a, y) \quad \text{Optional savings} \end{aligned}$$

  These functions are defined on discretised grids $a \in \mathcal{G}_a$ and $y \in \mathcal{G}_y$.

Iteration and backwards induction

Two main types of household problems:

Infinite-horizon problems

- Need to start with a guess for the solution; often this is just $V_0(a, y) = 0$
- Iterate on some object until consecutive iterations V_n, V_{n+1} are sufficiently close
- We can iterate either on value functions (VFI) or policy functions (PFI, EGM: endogenous grid-point method)

Finite-horizon problems

- Life-cycle and OLG models
- Solve for last period T
- Use backward induction to solve previous periods $T - 1, T - 2, \ldots$

Infinite horizon vs. life-cycle

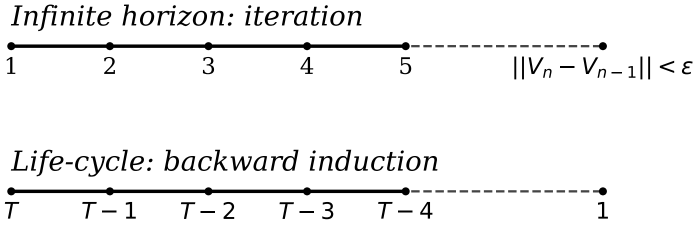

Figure 1: Solving infinite-horizon vs. life-cycle models

Overview of unit 3

Outline of remaining mini-lectures

- We exclusively solve household problems
  - Ignore distribution of households
  - Ignore general equilibrium
- Next mini-lectures:
  1. Lecture 1: Value function iteration (VFI)
     - Grid search
     - Interpolation
  2. Lecture 2: Endogenous grid-point method (EGM)
- Slides and pre-recorded lectures: general concepts, algorithms, results
- Live sessions: implement examples discussed in slides
- Hands-on approach to complement units 1–2

Overview of unit 3

Source code

- Github repository: https://github.com/richardfoltyn/mres-macro-topics
- Python and Matlab source code for examples discussed in lectures / live sessions
- We use Matlab in live sessions

Topics in Macroeconomics

Value function iteration (VFI)

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Topics covered in this unit

Solving household problems with VFI

We discuss the following models:

1. No uncertainty, no labour income: analytical solution
2. Certain labour income
3. Risky labour income

In all cases we assume CRRA preferences!

Solution methods

1. Grid search: no interpolation
2. "Unrestricted" maximisation:
    1. Linear interpolation
    2. Spline interpolation

Contents

VFI with analytical solution

Analytical solution

- Under some conditions value function has closed-form solution
- Assumptions:
  1. CRRA preferences
  2. No labour income
  3. No uncertainty
- Solution methods:
  1. Iteration on value function (“manual” VFI)
  2. Guess and verify (not covered)

Illustration: manually iterate on closed form, compare with numerical solution.

Household problem

Analytical VFI

Consider infinite-horizon consumption-savings problem with log preferences:

$$
\begin{aligned}
V(a) = \max_{c,\, a'} \; & \left\{ \log(c) + \beta V(a') \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a \\
& c \geq 0, \;\; a' \geq 0
\end{aligned}
$$

where

- a Beginning-of-period assets
- a' Next-period assets (savings)
- r Constant interest rate
- β Discount factor $\beta \in (0, 1)$

How can we find V using VFI?

- Initial guess:
  - Consume everything: $c = (1 + r)a$
  - Continuation value is zero
- Value function in iteration 1:

$$V_1(a) = \log(c) = \log\big((1+r)a\big) = \log(1+r) + \log(a)$$

- HH problem in iteration 2:

$$\begin{aligned} V_2(a) &= \max_{c,\, a'} \left\{ \log(c) + \beta V_1(a') \right\} \\ &= \max_{c,\, a'} \left\{ \log(c) + \beta \Big[\log(1+r) + \log(a') \Big] \right\} \\ \text{s.t.} \quad & c + a' = (1+r)a \\ & c \geq 0, \;\; a' \geq 0 \end{aligned}$$

Iteration 2

- First-order conditions:

$$c^{-1} = \lambda \qquad\qquad \beta\,(a')^{-1} = \lambda$$

  where λ is Lagrange multiplier on budget constraint.
- Eliminate Lagrange multiplier: $a' = \beta c$
- Substitute for a' in budget constraint to find policy functions:

$$c + \beta c = (1+r)a \implies c = (1+\beta)^{-1}(1+r)a$$
$$\implies a' = \beta(1+\beta)^{-1}(1+r)a$$

- Plug policy functions into value function:

$$V_2(a) = \log\left((1+\beta)^{-1}(1+r)a\right) + \beta\left[\log(1+r) + \log\left(\beta(1+\beta)^{-1}(1+r)a\right)\right]$$
$$= \beta\log\beta - (1+\beta)\log(1+\beta) + (1+2\beta)\log(1+r) + (1+\beta)\log(a)$$

After 2 iterations we have:

$$V_1(a) = \underbrace{\log(1+r)}_{\chi_1} + \underbrace{1}_{\varphi_1} \times \log(a)$$

$$V_2(a) = \underbrace{\beta \log \beta - (1+\beta)\log(1+\beta) + (1+2\beta)\log(1+r)}_{\chi_2} + \underbrace{(1+\beta)}_{\varphi_2} \log(a)$$

- Continue iterating? — Expressions become too complicated!
- Instead conjecture that value function takes the form

$$V_n(a) = \chi_n + \varphi_n \log(a) \tag{1}$$

We have shown this to be true for $n = 1, 2$

Solving via induction

Analytical VFI

- Assume V has functional form given in (1) for some n
- Then V_{n+1} will be given by

$$V_{n+1}(a) = \chi_{n+1} + \varphi_{n+1} \log(a)$$

- Task: find $\chi_{n+1}, \varphi_{n+1}$ given χ_n, φ_n
- We do this by solving

$$\begin{aligned} V_{n+1}(a) &= \max_{c,\, a'} \left\{ \log(c) + \beta V_n(a') \right\} \\ &= \max_{c,\, a'} \left\{ \log(c) + \beta \left[\chi_n + \varphi_n \log(a') \right] \right\} \\ \text{s.t.} \quad & c + a' = (1+r)a \\ & c \geq 0, \;\; a' \geq 0 \end{aligned}$$

Iteration $n+1$

- First-order conditions for the $(n+1)$-th iteration:

$$c^{-1} = \lambda \qquad\qquad \beta\varphi_n\,(a')^{-1} = \lambda$$

- Substitute for a' in budget constraint to find policy functions:

$$c + \beta\varphi_n c = (1+r)a \implies c = (1+\beta\varphi_n)^{-1}(1+r)a \tag{2}$$

$$\implies a' = \frac{\beta\varphi_n}{1+\beta\varphi_n}(1+r)a \tag{3}$$

- Plug policy functions into value function:

$$V_{n+1}(a) = \log\Big((1+\beta\varphi_n)^{-1}(1+r)a\Big) + \beta\left[\chi_n + \varphi_n \log\left(\frac{\beta\varphi_n}{1+\beta\varphi_n}(1+r)a\right)\right]$$

- Collect terms:

$$V_{n+1}(a) = \underbrace{\beta\chi_n + \beta\varphi_n \log(\beta\varphi_n) - (1+\beta\varphi_n)\Big[\log(1+\beta\varphi_n) + \log(1+r)\Big]}_{\chi_{n+1}} + \underbrace{(1+\beta\varphi_n)}_{\varphi_{n+1}} \log(a)$$

- Pin down sequence of $(\varphi_n)_{n=1}^{\infty}$:
  - Follows first-order linear difference equation

$$\varphi_{n+1} = 1 + \beta\varphi_n$$

  - General solution (φ_0 pinned down by $\varphi_1 = 1$):

$$\varphi_n = \beta^n \left(\varphi_0 - \frac{1}{1-\beta}\right) + \frac{1}{1-\beta}$$

  - Convergence to limit:

$$\varphi = \lim_{n\to\infty} \varphi_n = \frac{1}{1-\beta}$$

- Difference equation for χ_n is much more complicated (depends on φ_n!)
- Compute only limiting value (move $\beta\chi_n$ to l.h.s.):

$$\lim_{n\to\infty} \chi_{n+1} - \beta\chi_n = \lim_{n\to\infty} \beta\varphi_n \log(\beta\varphi_n) - (1+\beta\varphi_n)\Big[\log(1+\beta\varphi_n) + \log(1+r)\Big]$$

- Limiting value given by:

$$\chi = \lim_{n\to\infty} \chi_n = \frac{\beta}{1-\beta}\log\beta + \log(1-\beta) + \frac{1}{1-\beta}\log(1+r)$$

- Converged value function V:

$$V(a) = \chi + \frac{1}{1-\beta}\log(a)$$

Value function convergence

Analytical VFI

Convergence of coefficients χ_n and φ_n in

$$V_n(a) = \chi_n + \varphi_n \log(a)$$

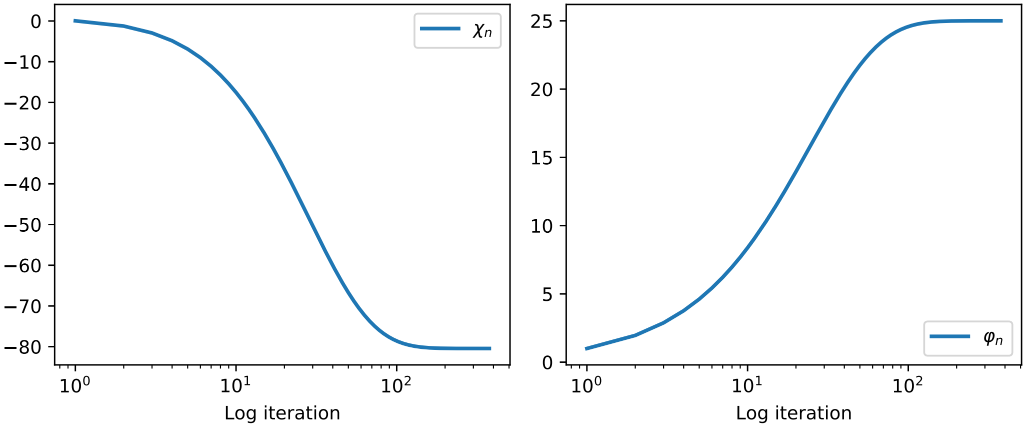

Figure 1: Convergence of analytical value function coefficients.

Analytical vs. numerical iteration

Analytical VFI

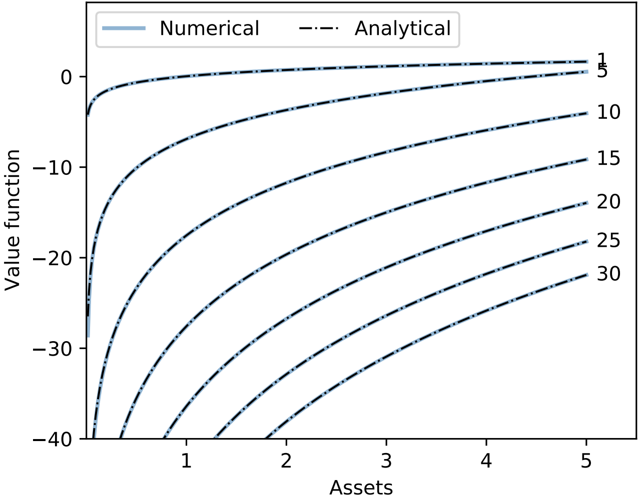

Figure 2: Value function V_n for the first few iterations.

Apply same reasoning to policy functions

- Define MPC as $\kappa_n \equiv (1 + \beta\varphi_n)^{-1}$
- Rewrite policy functions (2) and (3) at iteration $n + 1$ as:

$$\begin{aligned} c_{n+1} &= \kappa_n(1 + r)a \\ a'_{n+1} &= (1 - \kappa_n)(1 + r)a \end{aligned}$$

- $\lim_{n\to\infty} \kappa_n = 1 - \beta$
- Policy functions usually converge faster than value functions!

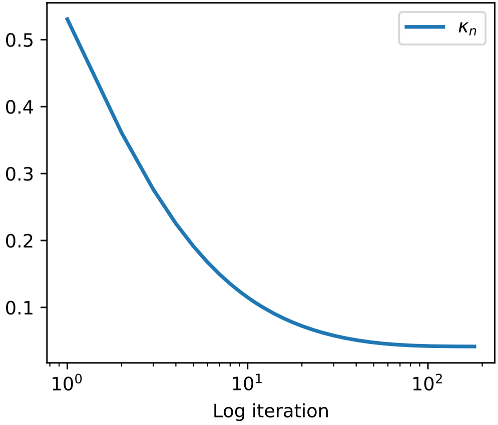

Figure 3: Convergence of policy function coefficient.

Analytical vs. numerical solution

Analytical VFI

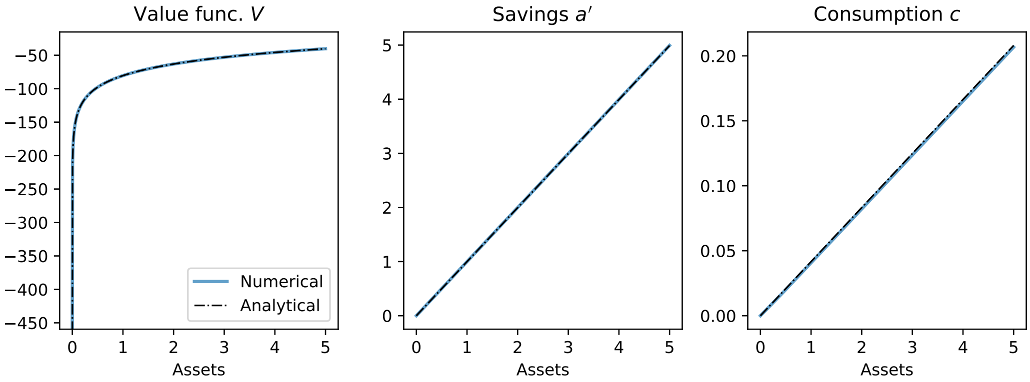

Figure 4: Converged value and policy functions.

VFI with grid search

VFI with grid search

- Restriction: solution method forces next-period assets to be exactly on discretized grid: $a' \in \mathcal{G}_a$
- Advantages:
  1. Easy to implement
  2. Derivative-free method
  3. Fast (unless grid is very dense)
- Disadvantages:
  1. Imprecise
  2. Policy functions are not smooth (unless grid is very dense)
  3. Does not scale well to multiple dimensions

Example: HH problem with constant labour income

VFI with grid search

- Infinitely-lived HH solves consumption-savings problem

$$
\begin{aligned}
V(a) = &\max_{c,\, a' \in \mathcal{G}_a} \Big\{ u(c) + \beta V(a') \Big\} \\
\text{s.t.} \quad & c + a' = (1+r)a + y \\
& c \geq 0, \;\; a' \geq 0
\end{aligned}
$$

  where

  $\mathcal{G}_a$ Beginning-of-period asset grid
  y Constant labour income

- Preferences are assumed to be CRRA with relative risk aversion γ:

$$
u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}
$$

- Note that with $\gamma = 1$, $u(c) = \log(c)$ as before

Solution algorithm

VFI with grid search

1. Create asset grid $\mathcal{G}_a = (a_1, \ldots, a_{N_a})$
2. Pick initial guess for value function, V_0
3. In iteration n, perform the following steps
    1. For each asset level a_i, find all feasible next-period asset levels $a_j \leq (1+r)a_i + y$, $a_j \in \mathcal{G}_a$
    2. For each j, compute consumption $c_j = (1+r)a_i + y - a_j$
    3. For each j, compute utility
    $$U_j = u(c_j) + \beta V_n(a_j) \tag{4}$$
    4. Find the index k that maximises (4):
    $$k = \arg\max_j \left\{ u(c_j) + \beta V_n(a_j) \right\}$$
    5. Set $V_{n+1}(a_i) = U_k$ and store k as the optimal choice at a_i

Parametrisation for problem with constant labour income

VFI with grid search

- The next slides show solutions for the following parametrisation:

	Description	Value
β	Discount factor	0.96
σ	Coef. of relative risk aversion	2
r	Interest rate	0.04
y	Labour income	1
N_a	Asset grid size	50, 100, 1000

Table 1: Parameters for HH problem with constant labour income

- Each graph compares three solution methods:
  1. VFI with grid search
  2. VFI with linear interpolation
  3. VFI with cubic spline interpolation

Solution for $N_a = 50$

VFI with grid search

Grid search is quite sensitive to grid size!

Compare results for $N_a = 50$, $N_a = 100$ and $N_a = 1000$.

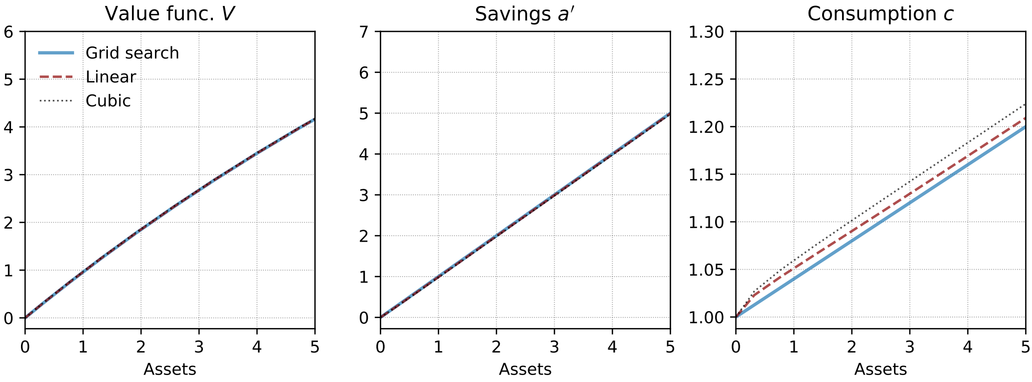

Figure 5: Solution with 50 asset grid points.

Solution for $N_a = 100$

VFI with grid search

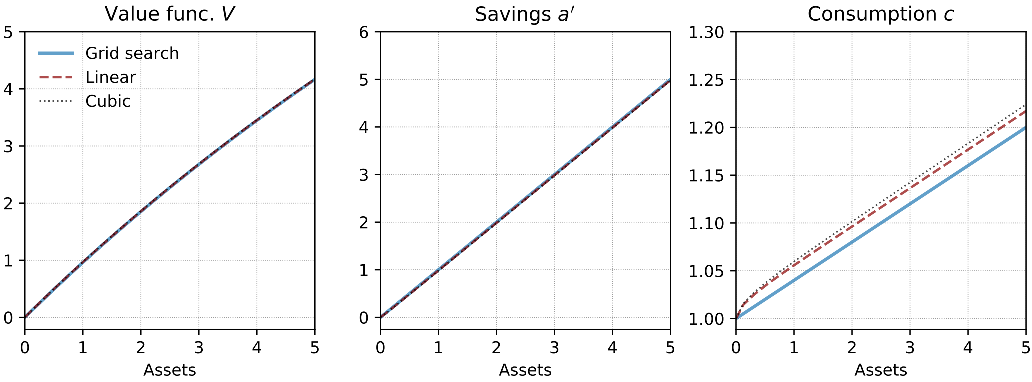

Figure 6: Solution with 100 assets grid points.

Solution for $N_a = 1000$

VFI with grid search

Figure 7: Solution with 1000 assets grid points.

VFI with interpolation

VFI with interpolation

- Grid search is rarely used today
- We prefer solution algorithms which find local maximum for each point on the grid (i.e. solution satisfies first-order conditions)
- Optimal points need not be on the grid, hence we have to interpolate
- Advantages:
  1. "Exact" solution (in a numerical sense)
  2. Less affected by curse of dimensionality in case of multiple choice variables
  3. Easier to spot mistakes since policy functions don't have artificial kinks as in grid search
- Disadvantages:
  1. Likely slower than grid search
  2. More complex to implement:
     - Need maximisation or root-finding routine
     - Need to compute derivatives of objective function or first-order condition, unless we use derivative-free methods or numerical differentiation.

Example: HH problem with risky labour income

VFI with interpolation

- Illustrate VFI with interpolation using standard Bewley/Huggett/Aiyagari problem with risky labour income
- Infinite-lived HH solves consumption-savings problem

$$V(a,y) = \max_{c,\, a'} \left\{ u(c) + \beta \mathbf{E}\left[V(a',y') \,\middle|\, y \right] \right\}$$

$$\text{s.t.} \quad c + a' = (1+r)a + y$$

$$c \geq 0, \;\; a' \geq 0$$

where

y Labour income process on state space $\mathcal{G}_y$ with transition probability $\Pr\left(y' = y_j \,|\, y = y_i \right) = \pi_{ij}$

- As before, $u(\bullet)$ is CRRA
- Note that now we have a two-dimensional state space on $\mathcal{G}_a \times \mathcal{G}_y$.

Solution algorithm

VFI with interpolation

1. Create asset grid $\mathcal{G}_a = (a_1, \ldots, a_{N_a})$
2. Create discrete labour income process with states $\mathcal{G}_y = (y_1, \ldots, y_{N_y})$ and transition matrix Π_y
3. Pick initial guess for value function, V_0
4. In iteration n, perform the following steps
    1. For each point (a_i, y_j) in the state space, find

       $$a^\star = \underset{a' \in [0, x_{ij}]}{\arg\max} \left\{ u\left(x_{ij} - a'\right) + \beta \sum_{k=1}^{N_y} \pi_{jk} V_n\left(a', y_k\right) \right\}$$

       where $x_{ij} = (1+r)a_i + y_j$ is the cash at hand.
    2. Compute value at optimum,

       $$V^\star = u\left(x_{ij} - a^\star\right) + \beta \mathbf{E}\left[V_n\left(a^\star, y'\right) \middle| y_j \right]$$

    3. Set $V_{n+1}(a_i, y_j) = V^\star$ and store $A_{n+1}(a_i, y_j) = a^\star$ as the savings policy function.

How do we find $a^\star$?

1. We use a maximiser that finds the maximum $a^\star \in [0, x_{ij}]$ of the function

$$f\left(a' \mid a_i, y_j\right) = u\left(x_{ij} - a'\right) + \beta \sum_{k=1}^{N_y} \pi_{jk} V_n\left(a', y_k\right)$$

   for given (a_i, y_j).
   - Need to interpolate $V_n(\bullet, y_k)$ onto arbitrary a'
   - Need to either use derivative-free maximizer, or differentiate df/da' numerically
2. In principle, we could perform *root-finding* on the FOC

$$-u'\left(x_{ij} - a'\right) + \beta \sum_{k=1}^{N_y} \pi_{jk} dV_n\left(a', y_k\right)/da' = 0$$

   This is rarely done since we don't know $dV_n\left(a', y_k\right)/da'$ and fast root-finders *additionally* need the derivative of the FOC!

Parametrisation for problem with risky labour

VFI with interpolation

- Assume labour process follows AR(1),

$$y_{t+1} = \rho y_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma^2\right)$$

  which we discretise as a Markov chain using the Rouwenhorst (1995) or Tauchen (1986) methods.

- The next slides show solutions for the following parametrisation:

	Description	Value
β	Discount factor	0.96
σ	Coef. of relative risk aversion	2
r	Interest rate	0.04
ρ	Autocorrelation of AR(1) process	0.95
σ	Conditional std. dev. of AR(1) process	0.20
N_y	Number of states for Markov chain	3
N_a	Asset grid size	50, 100, 1000

Table 2: Parameters for HH problem with risky labour income

Solution for $N_a = 50$

VFI with interpolation

Solution for different income levels: low, middle, high

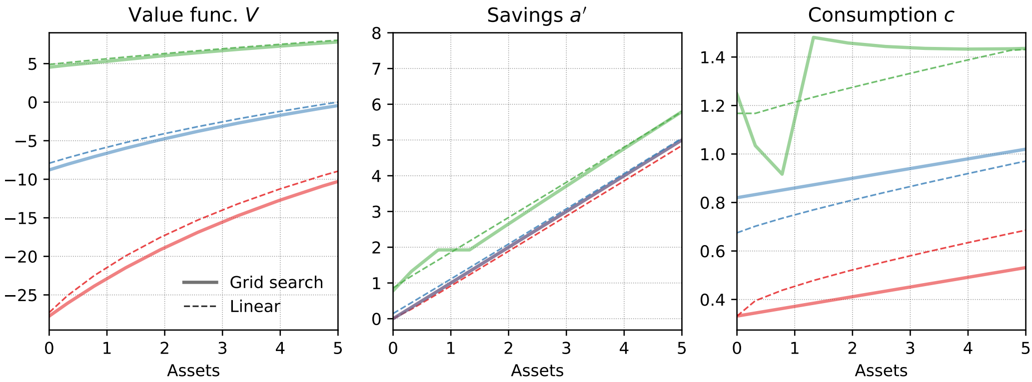

Figure 8: Solution with 50 asset grid points.

Solution for $N_a = 100$

VFI with interpolation

Solution for different income levels: low, middle, high

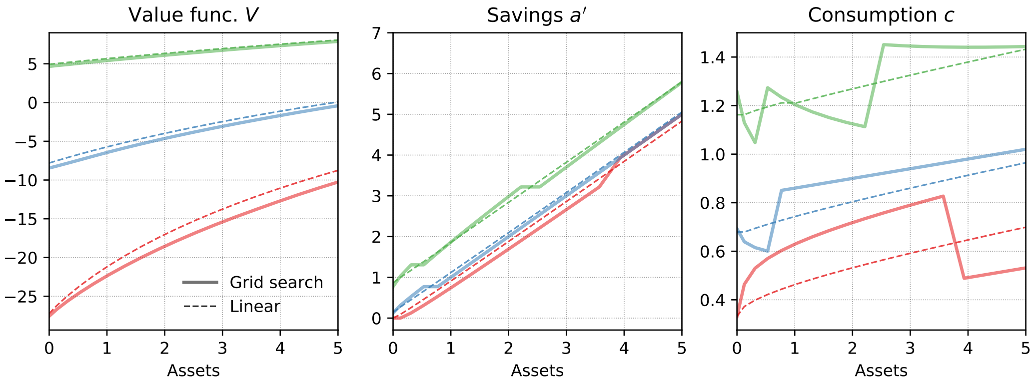

Figure 9: Solution with 100 assets grid points.

Solution for $N_a = 1000$

VFI with interpolation

Solution for different income levels: low, middle, high

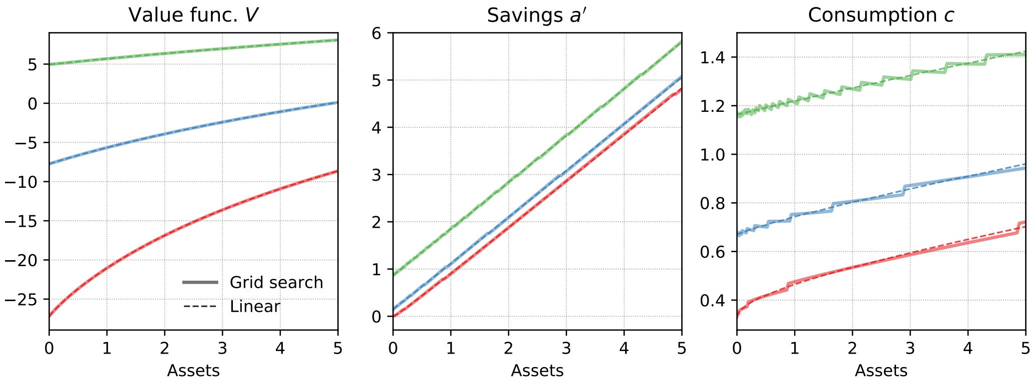

Figure 10: Solution with 1000 assets grid points.

Main take-aways

- Avoid grid search if you can!
- Test sensitivity of your solution to chosen grid size:
    - Check policy functions, value function almost always looks smooth!

Appendix:
Approximating AR(1) processes with Markov chains

AR(1) processes

Consider the following AR(1) process:

$$x_{t+1} = \mu + \rho x_t + \epsilon_{t+1} \qquad \epsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_\epsilon^2\right)$$

This process has the following conditional and unconditional moments:

	Conditional	**Unconditional**
Mean	$\mathbf{E}\left[\, x_{t+1} \mid x_t \,\right] = \mu + \rho x_t$	$\mathbf{E}\left[\, x_t \,\right] = \frac{\mu}{1-\rho}$
Variance	$\text{Var}\left(\, x_{t+1} \mid x_t \,\right) = \text{Var}\left(\, \epsilon_{t+1} \,\right) = \sigma_\epsilon^2$	$\text{Var}\left(\, x_t \,\right) = \frac{\sigma_\epsilon^2}{1-\rho^2}$
Autocorrelation	–	$\text{Corr}\left(x_{t+1}, x_t\right) = \rho$

Unconditional moments:

- Reflect long-run behaviour of a single process
- With a large cross-section of individuals, they also represent the cross-sectional mean and variance of the stationary distribution

Approximating AR(1) processes

- Any Markov chain approximation of an AR(1) needs to provide:
  1. The discrete state space $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$
  2. The transition matrix $\boldsymbol{\Pi}$ where the element (i, j) is the probability $\Pr\left(x_{t+1} = x_j \,|\, x_t = x_i \right)$

  Using these, we can find the ergodic (invariant, stationary) distribution $\boldsymbol{\lambda}$ over states $\boldsymbol{x}$ which satisfies

$$\boldsymbol{\lambda}' = \boldsymbol{\lambda}'\boldsymbol{\Pi}$$

- Approximation should match conditional / unconditional moments reasonably well!
- Frequently-used methods:
  1. Tauchen (1986)
  2. Rouwenhorst (1995): much better for processes with high persistence

Example: Income process

Assume that log income follows an AR(1) process:

$$\log y_{t+1} = \rho \log y_t + \epsilon_{t+1} \qquad \epsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_\epsilon^2\right)$$

with $\mu = 0$ (omitted), $\rho = 0.95$, $\sigma_\epsilon^2 = (0.2)^2$

Discretized Markov chain (Rouwenhorst method)

- State space in logs, transition matrix and ergodic distribution:

$$\log \boldsymbol{y} = \begin{bmatrix} -0.9058 \\ 0 \\ 0.9058 \end{bmatrix} \qquad \boldsymbol{\Pi} = \begin{bmatrix} 0.9506 & 0.0488 & 0.0006 \\ 0.0244 & 0.9512 & 0.0244 \\ 0.0006 & 0.0488 & 0.9506 \end{bmatrix} \qquad \boldsymbol{\lambda} = \begin{bmatrix} 0.25 \\ 0.50 \\ 0.25 \end{bmatrix}$$

- State space in levels:

$$\boldsymbol{y} = \begin{bmatrix} 0.4042 \\ 1.0000 \\ 2.4740 \end{bmatrix}$$

- Unconditional average income: $\mathrm{E} y_t = \boldsymbol{\lambda}' \boldsymbol{y} = 1.2195$

References I

Rouwenhorst, Geert K. (1995). “Asset Pricing Implications of Equilibrium Business Cycle Models”. In: **Frontiers of Business Cycle Research**. Ed. by Thomas F. Cooley. Vol. 10. Princeton University Press. Chap. 10, pp. 294–330.

Tauchen, George (1986). “Finite state markov-chain approximations to univariate and vector autoregressions”. In: **Economics Letters** 20.2, pp. 177–181.

Topics in Macroeconomics

Endogenous grid-point method (EGM)

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Motivation

Solving HH problems is often slow — Why?

- Consider standard infinite-horizon consumption-savings problem with states (a, y):
  - a Beginning-of-period assets
  - y Risky labour income following first-order Markov chain
- At each point (a_i, y_j) we maximise the objective

$$f(a') = u\left(x_{ij} - a'\right) + \beta \mathbf{E}\left[V(a', y') \middle| y_j \right]$$

  where x_{ij} is the cash at hand.
- Any numerical maximiser will call $f(\bullet)$ repeatedly to
  1. Determine the objective's value at some candidate point
  2. Determine the derivative at some candidate point
  3. Numerically differentiate the objective function
- This quickly adds up to numerous calls, which can be computationally expensive, depending on how difficult it is to compute expectations, etc.

Endogenous grid-point method

- The insight behind EGM (due to Carroll, 2006): Compute expectation only once!
- How can we do that if we don't know the optimal solution?
  - Exogenously impose the optimal solution (in the above case: a')
  - Determine implied beginning-of-period assets a
  - This gives rise to endogenous grid of beginning-of-period asset levels!

Endogenous grid-point method

Advantages

- Considerably faster than any other known method in this class of models
- No need for a maximiser or root-finder
- Works very well with *linear* interpolation, no need for splines, etc.

Disadvantages

- Does not always work
- Does not scale well to multiple continuous state or control variables (see Druedahl and Jørgensen (2017) for one attempted solution)
- Tricky (but possible) to combine with discrete choices, e.g. due to extensive-margin labour supply, fixed costs (see Iskhakov et al. (2017), Fella (2014))

Example: HH problem with risky labour

- Consider infinite-horizon consumption-savings problem

$$
\begin{aligned}
V(a,y) = \max_{c,\,a'} & \left\{ u(c) + \beta \mathbf{E}\left[V(a',y') \,\middle|\, y \right] \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a + y \\
& c \geq 0, \;\; a' \geq 0
\end{aligned}
$$

  where

  y Labour income process on state space $\mathcal{G}_y$ with transition probability $\Pr\left(y' = y_j \,|\, y = y_i \right) = \pi_{ij}$

- Preferences are CRRA:

$$
u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}
$$

Illustration of "standard" approach

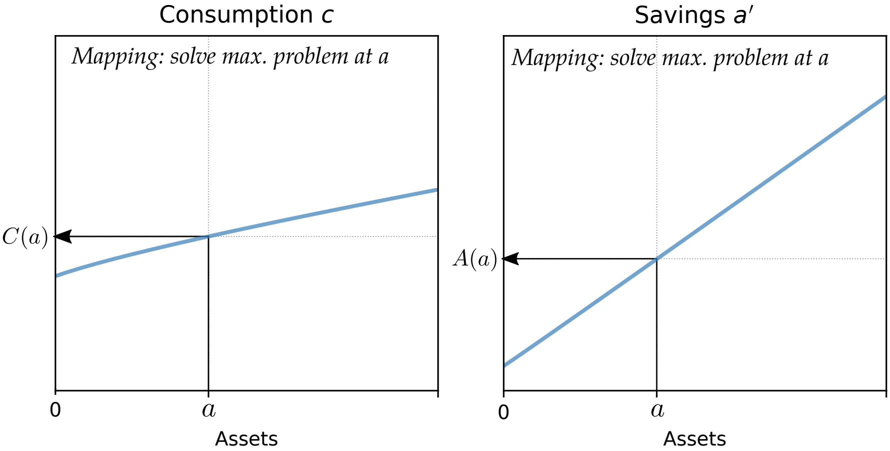

Figure 1: Mapping from exogenous assets to consumption and savings.

Illustration of EGM approach

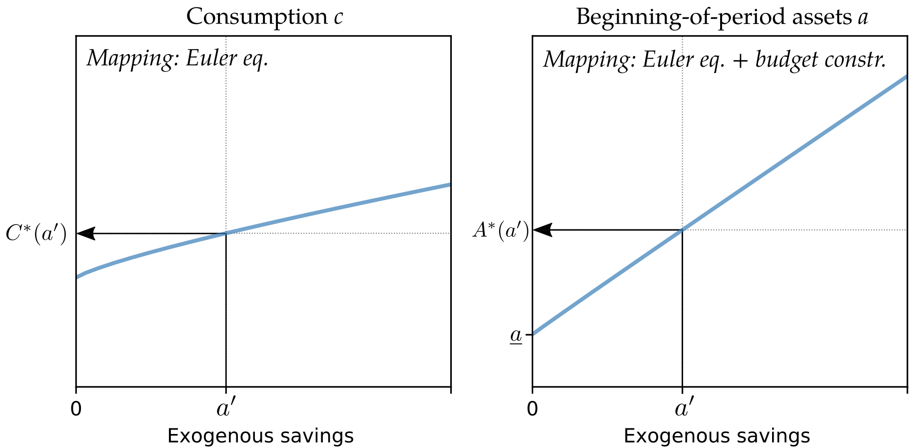

Figure 2: Mapping from exogenous savings to consumption and assets.

Deriving the Euler equation

- Combining the FOCs for c and a' yields the Euler equation

$$u'(c) = \beta \mathbf{E}\left[\partial V(a', y')/\partial a' \,\middle|\, y \right] \tag{1}$$

- For this problem, the envelope condition (see (5)) is

$$\frac{\partial V(a, y)}{\partial a} = (1 + r)u'\big(C(a, y)\big) \tag{2}$$

  where $C(a, y)$ is the consumption policy function.

- Combine (1) and (2) to get the more "familiar" variant of the Euler equation:

$$u'(c) = \beta(1 + r)\mathbf{E}\left[u'\big(C(a', y')\big) \,\middle|\, y \right]$$

Using the Euler equation

- Assume we know or have guessed $C(a', y')$
- We can exogenously fix a' and use $u'(c) = c^{-\gamma}$ to get an equation in a single unknown, c:

$$c = \left(\beta(1+r)\mathbf{E}\left[C(a', y')^{-\gamma} \middle| y \right]\right)^{-\frac{1}{\gamma}} \tag{3}$$

- From the BC, we can recover the implied beginning-of-period asset level a:

$$a = \frac{1}{1+r}\left[c + a' - y\right] \tag{4}$$

Solution to household problem

- To summarise, we found

  $c = C^*(a', y)$ Optimal consumption as a function of a'

  $a = A^*(a', y)$ Beginning-of-period assets as a function of a'

- Each a'_i gives us a tuple (a_i, c_i):
  - Use $(a_i, c_i)_{i=1}^{N_{a'}}$ to interpolate consumption policy onto exogenous beginning-of-period asset grid, $c = C(a, y)$
  - Use $\left(a_i, a'_i\right)_{i=1}^{N_{a'}}$ to interpolate savings policy onto exogenous beginning-of-period asset grid, $a' = A(a, y)$
- Important: using the Euler eq. implies that HH is at interior solution!
  - Implication: $\underline{a} = A^*(0, y)$ for $a' = 0$ is the highest asset level for which household does *not* save anything.
  - HH consumes everything for lower asset levels:

$$C(a, y) = (1 + r)a + y \qquad \forall\, a \leq \underline{a}$$

Solution algorithm (infinite horizon)

1. Fix exogenous savings grid $a' \in \mathcal{G}_{a'} = \left(a'_1, \ldots, a'_{N_{a'}}\right)$
2. Fix initial guess for consumption policy, $C_1(a, y)$. Usually the guess is to consume all resources.
3. In iteration n, proceed as follows:
    1. For each point (a'_i, y_j), compute the expectation
    $$m'_{ij} = \mathbf{E}\left[\, C_{n-1}(a'_i, y')^{-\gamma} \,\middle|\, y_j \,\right]$$
    2. Invert the Euler eq. as in (3) to get consumption today:
    $$c_{ij} = \left[\beta(1+r)m'_{ij}\right]^{-\frac{1}{\gamma}}$$
    3. Use the budget constraint as in (4) to find beginning-of-period assets:
    $$a_{ij} = \frac{1}{1+r}\left[c_{ij} + a'_i - y_j\right]$$
    4. Use the points (a_{ij}, c_{ij}) to get the updated consumption policy $C_n(\bullet, y_j)$ for each j. Set $C_n(a, y_j) = (1+r)a + y_j$ for all $a \leq \underline{a}_j$
4. Terminate iteration when C_{n-1} and C_n are close.

Solution algorithm (finite horizon)

1. Fix exogenous savings grid $a' \in \mathcal{G}_{a'} = \left(a'_1, \ldots, a'_{N_{a'}}\right)$
2. Compute consumption policy in terminal period T: this is usually $C_T(a, y) = (1 + r)a + y$, unless there is a bequest motive.
3. For each period $t = T - 1, T - 2, \ldots, 1$, proceed as follows:
   1. For each point (a'_i, y_j), compute the expectation
   $$m'_{ij} = \mathbf{E}\left[\, C_{t+1}(a'_i, y')^{-\gamma} \,\middle|\, y_j \,\right]$$
   2. Inver the Euler eq. as in (3) to get consumption today:
   $$c_{ij} = \left[\beta(1 + r)m'_{ij}\right]^{-\frac{1}{\gamma}}$$
   3. Use the budget constraint as in (4) to find beginning-of-period assets:
   $$a_{ij} = \frac{1}{1 + r}\left[c_{ij} + a'_i - y_j\right]$$
   4. Use the points (a_{ij}, c_{ij}) to get consumption policy $C_t(\bullet, y_j)$ for each j. Set $C_t(a, y_j) = (1 + r)a + y_j$ for all $a \leq \underline{a}_j$

- Assume labour process follows AR(1),

$$y_{t+1} = \rho y_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma^2\right)$$

  which we discretise as a Markov chain using the Rouwenhorst (1995) or Tauchen (1986) methods.

- The next slides show solutions for the following parametrisation:

	Description	Value
β	Discount factor	0.96
σ	Coef. of relative risk aversion	2
r	Interest rate	0.04
ρ	Autocorrelation of AR(1) process	0.95
σ	Conditional std. dev. of AR(1) process	0.20
N_y	Number of states for Markov chain	3

Table 1: Parameters for HH problem with risky labour income

Policy functions

EGM with linear interpolation

Solution for different income levels: low, middle, high

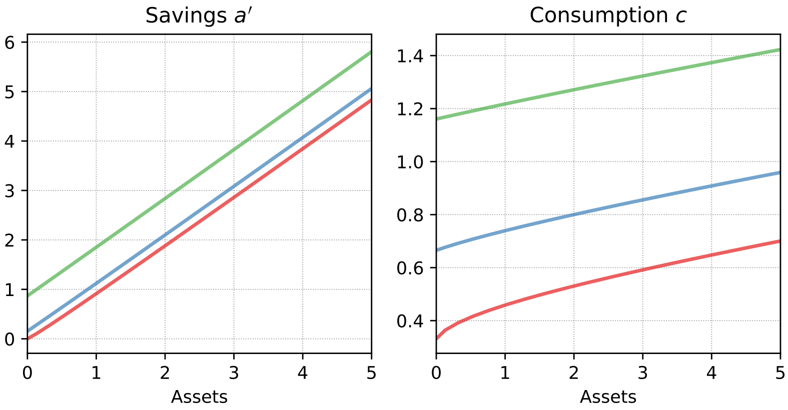

Figure 3: Solution with approx. 100 points on savings grid.

Solution for different income levels: low, middle, high

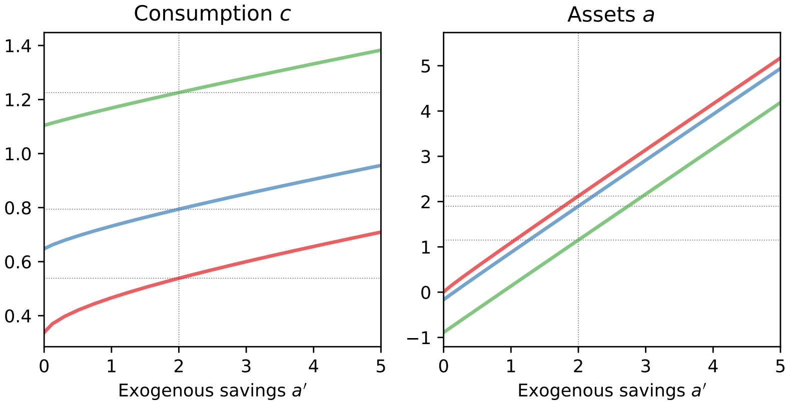

Figure 4: Solution with approx. 100 points on savings grid.

Relative run times

Run times for solving the above problem with $N_a = N_{a'} = 1000$ and $N_y = 3$:

Method	Time (seconds)	Rel. time
VFI – grid search	12.8	1.00
VFI – linear interpolation	170.6	13.32
EGM	0.4	0.03

When plain EGM fails

- Whenever we cannot determine where we "came from" (e.g. models with default)
- Discrete choices introduce jumps in policy functions:

Figure 5: Jumps due to discrete choice variables.

Main take-aways

Use EGM whenever you can!

- With only one continuous state, no discrete choices:
  - Straightforward application of plain EGM, potentially with minor extensions
  - Also includes models with portfolio choice, intensive-margin labour supply
- With additional discrete choice variables:
  - Probably works, but more tedious (e.g. Iskhakov et al. (2017))
  - Still considerably faster than VFI
- With multiple continuous state variables:
  - Probably not worth the effort

Appendix

Envelope condition

- Consider the following value function, where $a^\star$ are optimal savings $a^\star = A(a, y)$:

$$V(a,y) = u\left((1+r)a + y - a^\star\right) + \beta \mathbf{E}\left[V(a^\star, y) \middle| y \right]$$

We used the BC to substitute for $c^\star = (1+r)a + y - a^\star$

- Take derivatives w.r.t. a:

$$\begin{aligned} \frac{\partial V(a,y)}{\partial a} &= u'\left((1+r)a + y - a^\star\right)\left[(1+r) - \frac{\partial a^\star}{\partial a}\right] + \beta \mathbf{E}\left[\frac{\partial V(a^\star, y)}{\partial a^\star} \frac{\partial a^\star}{\partial a} \middle| y \right] \\ &= u'\left(c^\star\right)(1+r) + \frac{\partial a^\star}{\partial a} \underbrace{\left\{ -u'\left(c^\star\right) + \beta \mathbf{E}\left[\frac{\partial V(a^\star, y)}{\partial a^\star} \middle| y \right] \right\}}_{=0} \end{aligned} \tag{5}$$

- The FOC implies that the second term on the r.h.s. is zero!

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