Unit 1: Heterogeneity in Macroeconomics

Advanced Macroeconomics (ECON4040) – Part 2

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Heterogeneity in macroeconomics

So far, you studied representative-agent (RA) models: single household, single firm

When is the RA assumption justified? When is it problematic?

- ✓ Only interested in aggregate outcomes (quantities, prices)
- ✓ Economy aggregates (distribution of households is irrelevant)
- ✗ Economy does not aggregate: aggregate quantities or prices depend on distribution of households
- ✗ Interested in distribution as such (e.g., to study inequality)

- In this part of the course, we will be concerned with the last two cases
- This is in line with a gradual move towards heterogeneous-agent models in macroeconomics that started in the 1990s:
  - Example: RANK $\rightarrow$ TANK $\rightarrow$ HANK models for monetary policy analysis

Course outline

Teaching week 6 (Feb 13–17)

- Lecture: Unit 1: Introduction to heterogeneity in macro & inequality in the data

Teaching week 7 (Feb 20–24)

- Seminar: Exercises presented by group 1
- Lecture: Unit 2: Consumption over the lifecycle

Teaching week 8 (Feb 27–Mar 3)

- Seminar: Exercises presented by group 2
- Lecture: Unit 3: Consumption under uncertainty — Complete markets

Teaching week 9 (Mar 6–10)

- Lecture: Unit 4: Consumption under uncertainty — Incomplete markets

Teaching week 10 (Mar 13–17)

- Seminar: Exercises presented by group 3
- Lecture: Unit 5: Overlapping generations models

Teaching week 11 (Mar 20–24)

- In-course exam on March 23, 6–8:30pm

Outline for today

Two-period model with borrowing

Two-period household problem

- Workhorse model used for remainder of the course:
  - Two-period consumption-savings problem
  - CRRA preferences
  - Exogenous labour supply
  - Endowment economy (no production)
  - Often in partial equilibrium (today: GE)
- **Household problem**

$$
\begin{aligned}
\max_{c_1,\, c_2,\, a_2} \quad & u(c_1) + \beta u(c_2) \\
\text{s.t.} \quad c_1 + a_2 &= a_1 + y_1 \\
c_2 &= (1+r)a_2 + y_2 \\
c_1 \geq 0,\ c_2 &\geq 0 \qquad (1)
\end{aligned}
$$

- We ignore non-negativity constraints (1) from now on
- $u(\bullet)$ assumed to be CRRA (constant relative risk aversion)

CRRA preferences

- Most frequently used preference class in macroeconomics
- Special case: logarithmic preferences
- Utility function given by

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases}$$

Note: in economics log *almost always denotes the natural logarithm!*

- Parameter γ is called the coefficient of relative risk aversion (RRA)

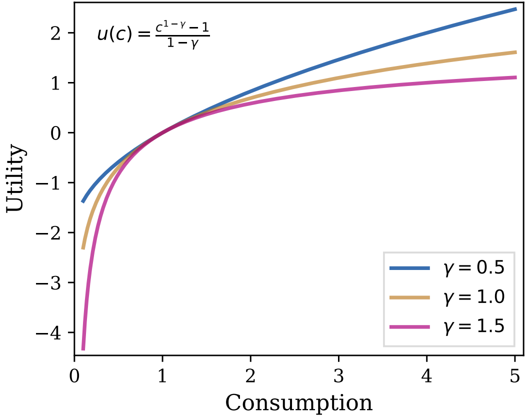


Figure 1: CRRA utility for different values of the relative risk aversion parameter γ

Two-period household problem **with** borrowing

- Simplifications for today:
  1. Log preferences: $u(c) = \log(c)$
  2. No discounting: $\beta = 1$

  More general setting covered in exercises and later units
- **Simplified two-period problem**

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \log(c_2) \tag{2}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1 \tag{3}$$

$$c_2 = (1 + r)a_2 + y_2 \tag{4}$$

- No restriction on a_2, household can save/lend ($a_2 > 0$) or borrow ($a_2 < 0$)
- Solution characterises optimal c_1, c_2 and a_2 as a function of parameters and exogenous quantities

Solving the problem: First-order conditions

Two-period household problem **with** borrowing

Consolidate per-period budget constraints into present-value lifetime budget constraint:

1. Substitute for a_2 in (4) using (3): $c_2 = (1+r)\big(y_1 - c_1\big) + y_2$
2. Divide by $1+r$, collect consumption on l.h.s., income on r.h.s.:

$$\underbrace{c_1 + \frac{c_2}{1+r}}_{\text{PV of cons.}} = \underbrace{y_1 + \frac{y_2}{1+r}}_{\text{PV of income}} \tag{5}$$

Lagrangian:

$$\mathcal{L} = \log(c_1) + \log(c_2) + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r}\right] \tag{6}$$

- $\lambda \geq 0$ is Lagrange multiplier for LTBC

First-order conditions (FOC): take derivatives w.r.t. c_1 and c_2

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{1}{c_2} - \frac{\lambda}{1+r} = 0 \tag{8}$$

Solving the problem: Euler equation

Two-period household problem **with** borrowing

We need to get rid of Lagrange multiplier λ. From FOCs (7) + (8) we have:

$$\lambda = \frac{1}{c_1} \qquad\qquad \lambda = (1+r)\frac{1}{c_2}$$

Eliminating λ, we get the Euler equation (EE):

$$\frac{1}{c_1} = (1+r)\frac{1}{c_2} \tag{9}$$

Interpretation

- Intertemporal optimality condition: household cannot do better by shifting consumption between periods 1 and 2.
- Could household do any better?
  1. Decrease consumption by one unit today, lose marginal utility $\frac{1}{c_1}$
  2. Save one unit, get $(1+r)$ units tomorrow
  3. Consumption tomorrow has marginal utility $\frac{1}{c_2}$ per unit, so household gains $(1+r)\frac{1}{c_2}$

  The Euler equation says that the household cannot be better off by doing this, so (9) has to hold!

Solving the problem: Optimal consumption

Two-period household problem **with** borrowing

Solve Euler equation (9) for $c_2 = (1+r)c_1$

Plug into lifetime budget constraint (5): optimal consumption in $t = 1$

$$c_1 + \frac{(1+r)c_1}{1+r} = y_1 + \frac{y_2}{1+r} \implies c_1 = \frac{1}{2}\left[y_1 + \frac{y_2}{1+r}\right] \tag{10}$$

Plug into EE to get optimal consumption in $t = 2$:

$$c_2 = \frac{1}{2}\Big[(1+r)y_1 + y_2\Big] \tag{11}$$

Solution to HH problem: allocation (c_1, c_2)

Interpretation

- No discounting, no borrowing constraints, hence optimal to consume half of lifetime income in each period

- So far we only solved partial equilibrium problem
- Interest rate r taken as given
- Need to specify income y_1, y_2 to solve for equilibrium r

Heterogeneous-agent economy with two households

- Households A and B have identical preferences, but different endowments:

$$y_1^A = 3,\ y_2^A = 1$$
$$y_1^B = 1,\ y_2^B = 3$$

- Will households want to consume their income each period? — No! (contradicts consumption smoothing)
- A and B trade to attain higher utility: A acts as lender, B as borrower in period 1

- What is a general equilibrium? — Interest rate r such that markets clear
- Markets in this economy:
  1. Goods market in period 1 (equivalent: market for savings)
  2. Goods market in period 2
- Solution approach: find r to clear one market, other one clears by Walras' law.

Example: goods market clearing in period 1

$$\underbrace{c_1^A + c_1^B}_{\text{Aggregate consumption}} = \underbrace{y_1^A + y_1^B}_{\text{Aggregate endowment}}$$

Equivalent to market for savings in period 1:

$$\underbrace{y_1^A - c_1^A}_{\text{Savings by } A} = \underbrace{c_1^B - y_1^B}_{\text{Borrowing by } B} \tag{12}$$

Derive equilibrium interest rate r from savings market clearing using (10):

$$
\begin{aligned}
y_1^A - c_1^A &= c_1^B - y_1^B \\
y_1^A - \frac{1}{2}\left[y_1^A + \frac{y_2^A}{1+r}\right] &= \frac{1}{2}\left[y_1^B + \frac{y_2^B}{1+r}\right] - y_1^B \\
y_1^A + y_1^B &= \frac{y_2^A + y_2^B}{1+r} \qquad (13)
\end{aligned}
$$

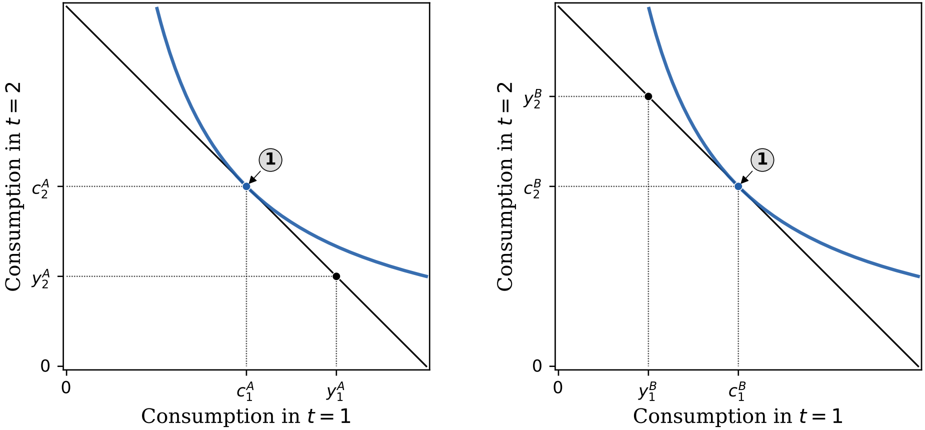
Define aggregate income in each period t: $Y_t = y_t^A + y_t^B$

Equilibrium interest rate follows from (13):

$$
Y_1 = \frac{Y_2}{1+r} \implies r = \frac{Y_2}{Y_1} - 1 \qquad (14)
$$

For our example we have: $Y_1 = Y_2 = 4 \implies r = 0$

Explain the intuition behind $r = 0$!



(a) Type A **(b)** Type B

Figure 2: General equilibrium in **with** borrowing. (1) shows the equilibrium allocation and the blue lines are the corresponding indifference curves.

Two-period model without borrowing

Two-period household problem **without** borrowing

- Previous example: A was lender, B was borrower
- What happens if we impose no-borrowing constraint?
- Household problem almost as before:

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \log(c_2) \tag{15}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = (1+r)a_2 + y_2$$

$$a_2 \geq 0 \tag{16}$$

New: Inequality constraint (16)

- Remaining environment unchanged:
  Two households, A and B, with income

$$y_1^A = 3,\ y_2^A = 1$$
$$y_1^B = 1,\ y_2^B = 3$$

Solution method

Two-period household problem **without** borrowing

Two possible approaches:

1. Shortcut exploiting economic intuition (and what we know from the previous example *with* borrowing)
2. Solve constrained maximisation problem with occasionally binding borrowing constraint

Previously we found:

- Type A saves in equilibrium (this is still possible)
- Type B borrows in equilibrium (no longer possible)

Solution method

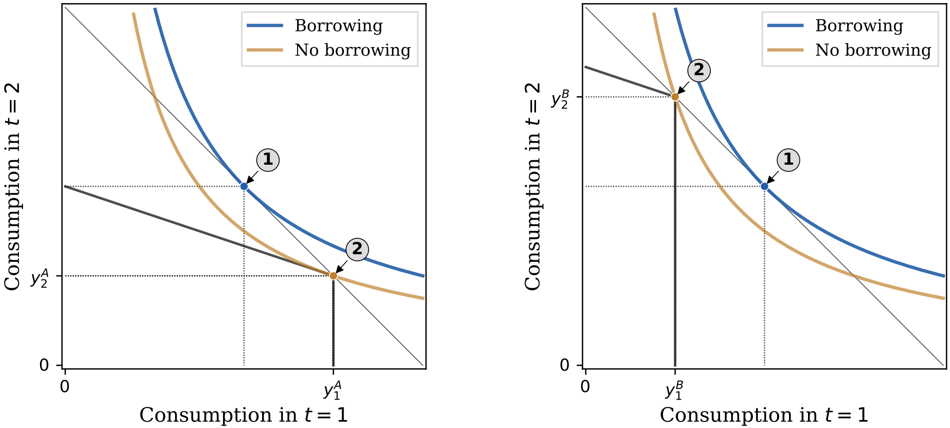
1. Type B cannot borrow $\Rightarrow$ consumes income each period
2. No borrowing $\Rightarrow$ in equilibrium no one can save because there is no counterparty (assets are in zero net supply)
3. Saving is permitted $\Rightarrow$ need to find equilibrium r such that A does not want to save

How to find equilibrium r?

- B's Euler equation does not hold (not an interior solution)
- Need to use A's Euler equation with $c_1^A = y_1^A$ and $c_2^A = y_2^A$:

$$\frac{1}{c_1^A} = (1+r)\frac{1}{c_2^A} \implies \frac{1}{y_1^A} = (1+r)\frac{1}{y_2^A} \implies r = \frac{y_2^A}{y_1^A} - 1$$

- Equilibrium interest rate: $r = \frac{1}{3} - 1 \approx -66.7\%$ **r is very low! Intuition?**



(a) Type *A* **(b)** Type *B*

Figure 3: General equilibrium **without** borrowing. (1) shows the unattainable allocation with borrowing, while (2) is the new autarky allocation. The thick black line depicts the budget line without borrowing, the blue line the indifference curve with borrowing, and the yellow line the indifference curve without borrowing.

Set up Lagrangian with inequality constraints. Several ways to do this:

1. Use lifetime budget constraint as in (6), impose $c_1 \leq y_1$ which implies $a_2 \geq 0$
2. Eliminate c_1 and c_2, leaving a_2 as the only choice; impose $a_2 \geq 0$
3. Use per-period budget constraints, impose $a_2 \geq 0$

Lagrangian for variant 3 (compare to unconstrained variant in (6)):

$$\mathcal{L} = \log(c_1) + \log(c_2) + \lambda_1 \underbrace{\Big[y_1 - a_2 - c_1\Big]}_{\text{Budget constr. } t=1} + \lambda_2 \underbrace{\Big[(1+r)a_2 + y_2 - c_2\Big]}_{\text{Budget constr. } t=2} + \lambda_a \cdot \underbrace{a_2}_{\text{Borrowing constr.}} \tag{17}$$

How to impose inequality constraints?
Example: want to impose $x \geq y$

1. Rewrite as $x - y \geq 0$
2. Add to Lagrangian as $\lambda(x - y)$ with Lagrange multiplier $\lambda \geq 0$

First-order conditions: take derivatives w.r.t. c_1, c_2, a_2:

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda_1 = 0 \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{1}{c_2} - \lambda_2 = 0 \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial a_2} = -\lambda_1 + \lambda_2(1+r) + \lambda_a = 0 \tag{20}$$

Complementary slackness condition: $\lambda_a \cdot a_2 = 0$

1. Constraint is binding $\Rightarrow a_2 = 0, \lambda_a \geq 0$
2. Constraint not binding $\Rightarrow a_2 > 0, \lambda_a = 0$

In both cases, $\lambda_a \cdot a_2 = 0$ holds!

Intuition

- Recall interpretation of Lagrange multiplier: change in objective if constraint is relaxed by 1
- If constraint is <u>not</u> binding, relaxing it does <u>not</u> change objective!

Euler equation: consolidate FOCs, eliminate λ_1, λ_2

$$\frac{1}{c_1} = (1+r)\frac{1}{c_2} + \lambda_a \tag{21}$$

We don't know r or λ_a — so how is this useful?

Approach: Guess and verify

Step 1: Guess

1. At equilibrium r, type B will be at constraint $\Rightarrow \lambda_a^B > 0$
   B's Euler equation is not helpful (too many unknowns)
2. A will not want to borrow $\Rightarrow \lambda_a^A = 0$
3. Savings in zero net supply, so both A and B have to consume their income: $c_1^A = y_1^A$, $c_2^A = y_2^A$, $c_1^B = y_1^B$, $c_2^B = y_2^B$

Solving the problem: Guess and verify

Two-period household problem **without** borrowing

Given our guess, $\lambda_a^A = 0$, so r follows from A's Euler equation:

$$\frac{1}{c_1^A} = (1+r)\frac{1}{c_2^A} \implies \frac{1}{y_1^A} = (1+r)\frac{1}{y_2^A} \implies r = \frac{y_2^A}{y_1^A} - 1$$

Equilibrium interest rate: $r = \frac{1}{3} - 1 \approx -66.7\%$

Step 2: Verify
Plug equilibrium r into B's Euler equation:

$$\frac{1}{c_1^B} = (1+r)\frac{1}{c_2^B} + \lambda_a^B \implies \frac{1}{1} = \left(1 - \frac{2}{3}\right)\frac{1}{3} + \lambda_a^B$$
$$\implies 1 = \frac{1}{9} + \lambda_a^B$$
$$\implies \lambda_a^B = \frac{8}{9} > 0$$

Household B is at borrowing constraint, as conjectured. ✓

Aggregation

Do economies from previous examples aggregate?

- Previous examples had heterogeneous agents (HA), A and B
- Assume we are only interested in aggregates:
  - Quantities: $C_t = c_t^A + c_t^B, \quad Y_t = y_t^A + y_t^B$
  - Prices: r
- Can we find representative-agent (RA) economy with a single household that generates these?

  Assumptions:

  1. RA has same preferences as A and B
  2. RA gets aggregate endowment $Y_t = y_t^A + y_t^B$:

$$
\begin{aligned}
Y_1 &= y_1^A + y_1^B = 3 + 1 = 4 \\
Y_2 &= y_2^A + y_2^B = 1 + 3 = 4
\end{aligned}
$$

Aggregation: Economy **with** borrowing

- RA solves the same maximisation problem, Euler equation same as in (9):

$$\frac{1}{C_1} = (1 + r^*)\frac{1}{C_2}$$

- No trade in equilibrium (no one to trade with!):

$$C_1 = Y_1 \qquad\qquad C_2 = Y_2$$

- Equilibrium interest rate r^* needs to satisfy Euler equation:

$$r^* = \frac{C_2}{C_1} - 1 = \frac{Y_2}{Y_1} - 1 = \frac{4}{4} - 1 = 0$$

Same expression as in (14) for heterogeneous-agent economy.

Conclusion: $r^* = r$, economy aggregates!

Aggregation: Economy **without** borrowing

- For the RA, nothing changed compared to scenario *with* borrowing.
  In particular, we still have $Y_1 = Y_2 = 4$
- Euler equation yields same equilibrium interest rates as before, $r^* = 0$
- Compare to HA economy: $r = -66.7\%$

Conclusion: $r^* \neq r$, economy does not aggregate!

Aggregation usually fails with incomplete markets, e.g.,

- Idiosyncratic risk that cannot be perfectly insured
- Borrowing constraints

Measures of inequality

Measures of inequality

Why do we need to quantify heterogeneity?

- Previous section: heterogeneity can matter for aggregates
- Heterogeneity interesting in itself (e.g., to study inequality)

Along which dimensions do we observe inequality in the data?

- Wealth
- Income, employment status
- Consumption, Leisure
- Age, health, life expectancy

Which inequality measures have you encountered so far?

- Gini coefficient (Lorenz curve)
- Variance of logs
- Percentile ratios: 90–10, 90–50, 50–10

- Measures distance from perfect equality:
  - Gini = 0: everyone has same amount
  - Gini = 1: everything is owned by one person or household
- Gini can be computed using size of areas A and B:

$$\mathcal{G} = \frac{A}{A+B} = 2A = 1 - 2B$$

- Example shown in figure:
  - Lower quartile owns 6%
  - Lower three quartiles own 56%

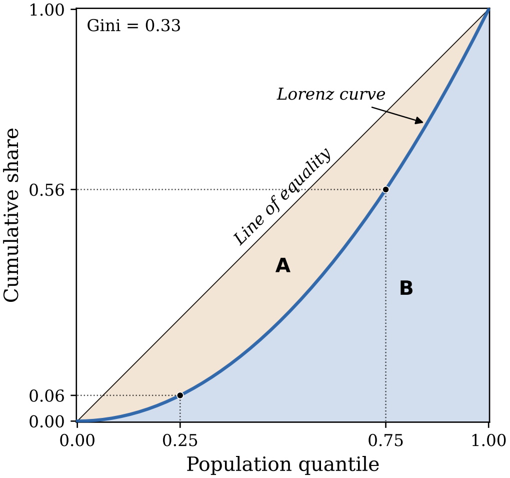


Figure 4: Lorenz curve and graphical representation of the Gini coefficient

Illustration of extreme cases: Gini = 0, Gini = 1

- Gini can exceed 1 if variable of interest can be negative (e.g., net worth)

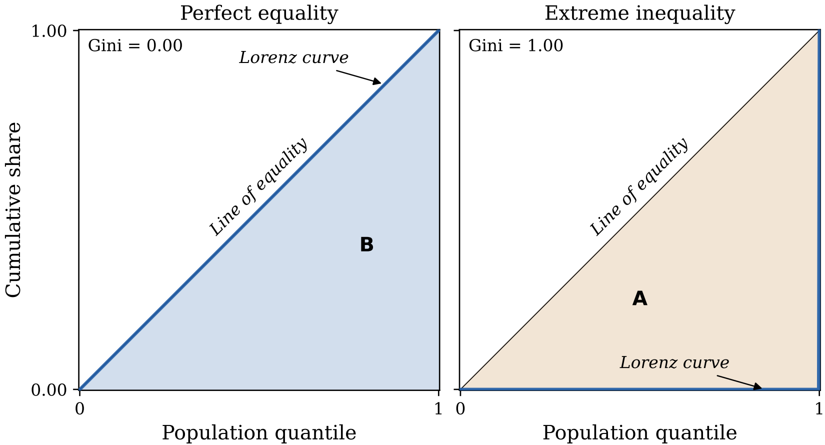


Figure 5: Lorenz curve and Gini for the extreme cases of "perfect" equality and inequality.

- Hypothetical income distribution in economy with 5 households:

HH	Income in $	Share	Cum. share
1	15,750	3.0%	3.0%
2	35,650	6.7%	9.7%
3	58,950	11.1%	20.8%
4	96,790	18.2%	39.0%
5	324,090	61.0%	100.0%

- Closely represents mean income by quintile in US (based on SCF)

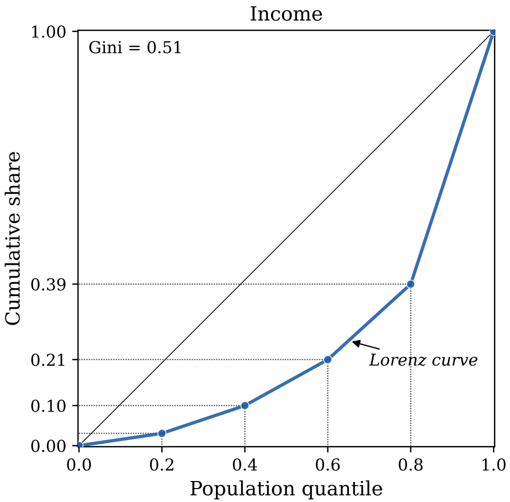


Figure 6: Lorenz curve and Gini for hypothetical income distribution

- Hypothetical wealth distribution in economy with 4 households:

HH	Wealth in \$	Share	Cum. share
1	−13,630	−0.5%	−0.5%
2	58,180	1.9%	1.5%
3	236,280	7.9%	9.4%
4	2,706,290	90.6%	100.0%

- Approximates mean net worth by quartile in US (based on SCF)

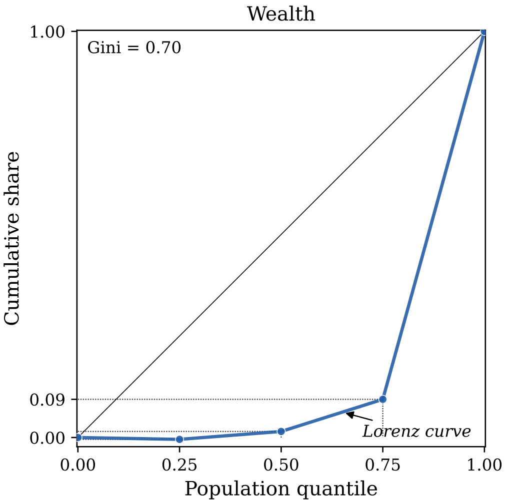


Figure 7: Lorenz curve and Gini for hypothetical wealth distribution

Why more than one?

- No unique or best way to summarise whole distribution in a single statistic
- Measures respond differently to inequality in different parts of the distribution

Other inequality measures

- Variance of logs: less sensitive to inequality at the top
- Percentile ratios: 90–10, 90–50, 50–10
  - Measure relative distance between two percentiles of a distribution
  - Example: if 90–10 ratio = 5, then household at 90^{th} percentile has five times more resources than household at 10^{th} percentile
- Allow us to zoom in on specific parts of the distribution
  - Example: movements in 50–10 tell us about changes in bottom half of distribution

Inequality in the US and UK

Inequality in the data

Which data would you collect to measure inequality?

We need micro data on individuals or households, not (aggregate) time series!

- Panel (longitudinal) data
- Cross-sectional data
- Rotating (short) panels

How would you rank inequality in wealth, gross income, disposable income, and consumption?

We usually observe the following ranking (in decreasing order):

1. Wealth
2. Gross income
3. Disposable income
4. Consumption

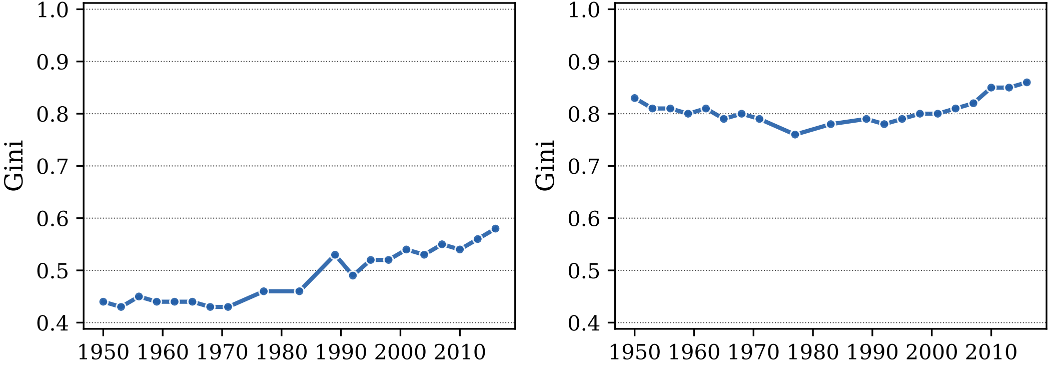
Public data sources for the US

- Current Population Survey (CPS)
- Panel Study of Income Dynamics (PSID)
- Health and Retirement Study (HRS)
- Survey of Consumer Finances (SCF)
- Consumption Expenditure Survey (CEX)

Data sets differ in variables they collect (consumption, income, wealth) and which samples they target (representative for the US, the elderly, etc.)

Inequality trends in the US

- Income Gini increased substantially (0.43 in 1971 to 0.58 in 2016)
- Less clear trend in wealth Gini



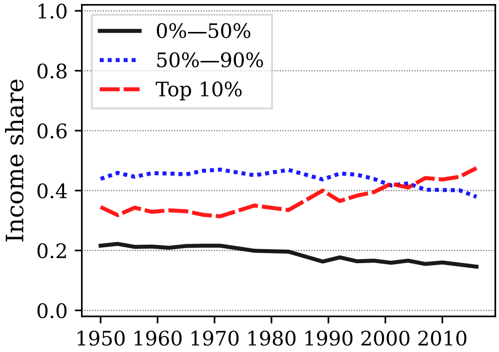
(a) Income

(b) Wealth

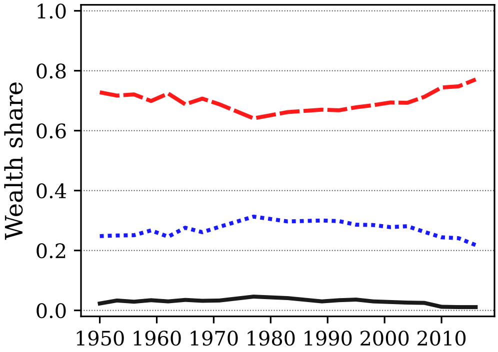
Figure 8: Gini for gross household income (including transfers) and household net worth in the US, 1950–2016. Data source: Kuhn, Schularick, and Steins (2020, Table E.5)

Income and wealth shares in the US

- Gini does not easily convey which parts of the distribution gained or lost
- Look at income and wealth shares instead!
- Top 10% increased income share from 36% to 48%



(a) Income

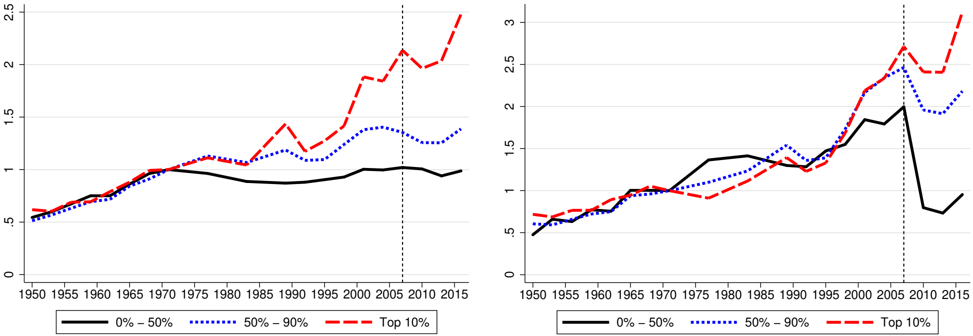


(b) Wealth

Figure 9: Shares of income and wealth in the US, 1950–2016. Data source: Kuhn, Schularick, and Steins (2020, Table E.4)

Income and wealth growth in the US

- Income growth diverged already in 1970s
- Wealth growth much more even until about 2000



(a) Income growth

(b) Wealth growth

Figure 10: Income and wealth growth for the bottom 50%, the middle class (50%–90%) and the top 10% of the wealth distribution. All time series are normalised to one in 1971. The dashed vertical line in 2007 shows the Great Recession. Source: Kuhn, Schularick, and Steins (2020, Figure 12)

Consumption inequality in the US

As economists, shouldn't we only care about consumption / leisure inequality?

- Consumption inequality is smaller: (in-kind) transfers, intra-family insurance, etc.
- Increase over last decades tracks rise in income inequality

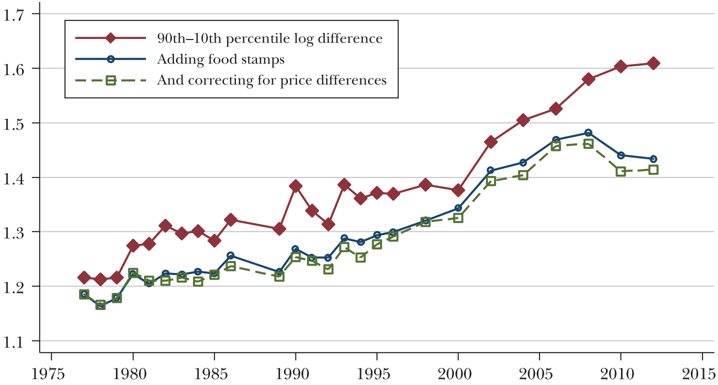


Figure 11: Difference between the 90th and the 10th percentiles of distribution of the logarithm of food consumption, 1977–2012. Source: Attanasio and Pistaferri (2016, Figure 2), based on PSID data.

Consumption inequality in durable goods (ownership rates)

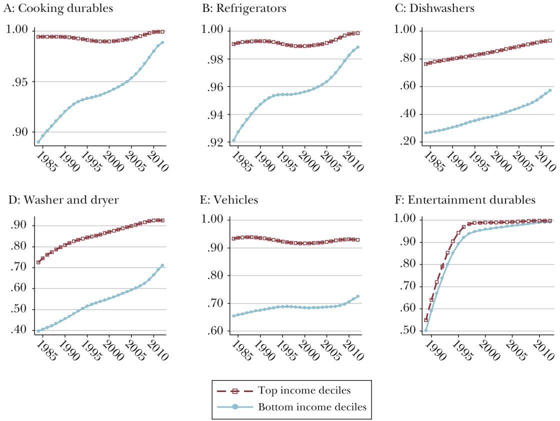


Figure 12: Ownership rates for selected durables for top and bottom after-tax income deciles. Source: Attanasio and Pistaferri (2016, Figure 3), based on CEX.

Leisure inequality in the US

- Can more leisure compensate for lower income or consumption?
- More leisure can be involuntary (e.g., unemployment)

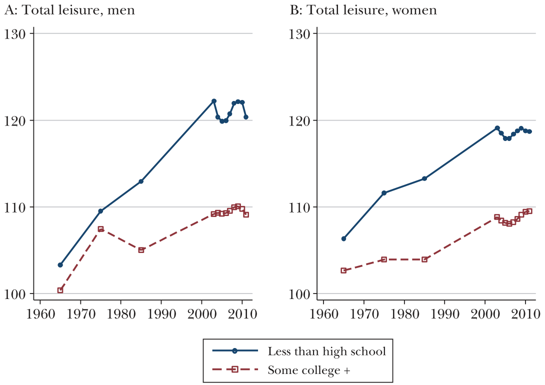


Figure 13: Total leisure hours per week, defined as the sum of social activities, active and passive leisure, and time devoted to personal care (which includes sleeping). Source: Attanasio and Pistaferri (2016, Figure 4), based on US time use data.

Data sets to study inequality in the UK

Public data sources for the UK

- British Household Panel Survey (BHPS)
- Understanding Society
- Labour Force Survey (LFS)
- Family Resources Survey (FRS)
- Living Costs and Food Survey (LCF)

Income inequality in the UK

- Upward trend in 1970s and 1980s similar to US
- Broadly constant thereafter, or even decreasing in bottom 90%

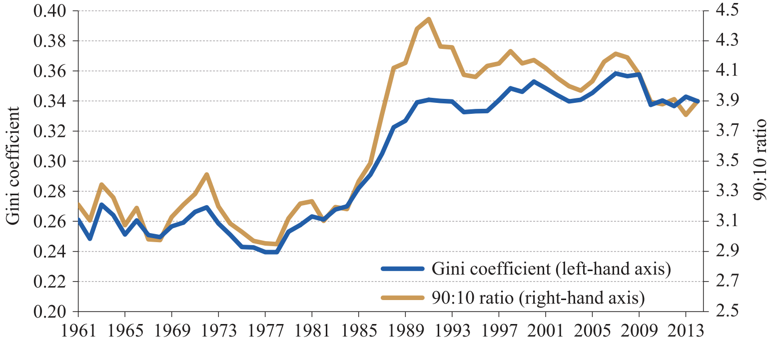


Figure 14: The Gini coefficient and the 90-10 ratio of net household income (adjusted for household size) in Great Britain, 1961–2014. Source: Belfield et al. (2017, Figure 2)

Income inequality in the UK

Income inequality from gross income to disposable income: illustrates redistributive tax/transfer system.

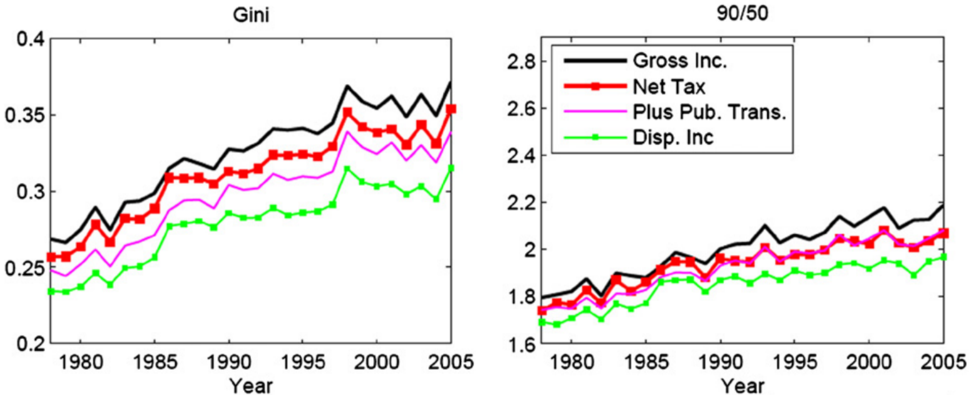


Figure 15: Change in inequality when moving from gross income to disposable income. Source: Blundell and Etheridge (2010, Figure 4.4), based on FES data

Main takeaways from this unit

Main takeaways

Models / theory

We introduced the following concepts:

1. Heterogeneous agents (HA) in general equilibrium models
2. Borrowing constraints, constrained optimisation
3. Aggregation: can representative-agent (RA) model replicate aggregate quantities and prices of HA model?

Inequality in the data

1. Inequality measures: Gini coefficient, variance of logs, percentile ratios
2. Inequality ranking: wealth $\succ$ income $\succ$ consumption
3. Redistributive taxes and transfers mitigate inequality: gross income $\succ$ disposable income
4. Income inequality increased over last five decades, more so in the US than the UK

References

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Unit 2: Consumption over the Life Cycle

Advanced Macroeconomics (ECON4040) – Part 2

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February 24, 2023

Outline for today

1. Consumption responses to changes in interest rate
   - Income, substitution and wealth effects
   - Elasticity of intertemporal substitution
2. Life cycle models with many periods
3. Life cycle profiles in the data
4. Main takeaways

Substitution, income and wealth effects

Income and substitution effects with log preferences

Model environment

- Two-period consumption-savings problem
- Log preferences
- No income in period 2 (we relax this below)
- Partial equilibrium (exogenous r)
- Household solves:

$$\max_{c_1,\ c_2,\ a_2} \quad \log(c_1) + \beta \log(c_2)$$
$$\text{s.t.} \quad c_1 + a_2 = a_1 + y_1$$
$$c_2 = (1+r)a_2$$

Want to answer the following:

- How do optimal (c_1, c_2) respond to changes in r?
- How to decompose total response into income and substitution effect?

Solving the problem: Rinse/repeat from unit 1

Log preferences, no period-2 income

1. Lifetime budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 \tag{1}$$

2. Lagrangian:

$$\mathcal{L} = \log(c_1) + \beta \log(c_2) + \lambda \left[y_1 - c_1 - \frac{c_2}{1+r} \right]$$

3. First-order conditions for c_1, c_2:

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta \frac{1}{c_2} - \lambda \frac{1}{1+r} = 0 \tag{3}$$

3. Euler equation: (2) + (3)

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2} \tag{4}$$

4. Optimal consumption: (1) + (4)

$$c_1 = \frac{1}{1+\beta} y_1 \tag{5}$$

$$c_2 = \frac{\beta}{1+\beta}(1+r)y_1 \tag{6}$$

Consumption response to changes in r

Log preferences, no period-2 income

How does c_1 in (5) respond to changes in r? — Not at all, does not depend on r!

Why? — Income and substitution effects cancel for log preferences

Substitution effect

- Change in demand as relative price changes while keeping utility level constant

Income effect

- Often defined as the *residual* after accounting for SE
- Depends on net asset position:
  - Lender: interest rate $\uparrow \implies$ interest income $\uparrow$
  - Borrower: interest rate $\uparrow \implies$ cost of borrowing $\uparrow$
- Consumption in both periods are normal goods, hence:
  - Household gets richer $\implies c_1, c_2 \uparrow$
  - Household gets poorer $\implies c_1, c_2 \downarrow$

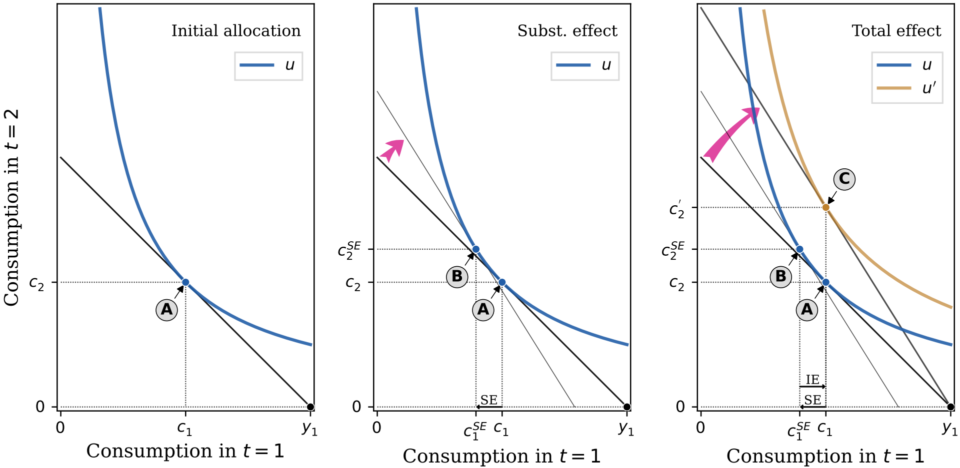


Figure 1: Income and substitution effects of an increase in r for a <u>lender</u> with <u>log preferences</u> and no second-period income

In previous example, household received all income in first period.

How would our findings change with income in period 2?

Wealth effect: Present value of income in later periods responds to changes in r

- Even with log preferences, change in r affects consumption c_1

Illustration with income in both periods

- Household solves:

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \beta \log(c_2)$$
$$\text{s.t.} \quad c_1 + a_2 = a_1 + y_1$$
$$c_2 = (1+r)a_2 + y_2$$

New: Receives income (y_1, y_2) in both periods

Solving the problem: Rinse/repeat from unit 1

Log preferences and period-2 income

1. Lifetime budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{7}$$

2. Lagrangian:

$$\mathcal{L} = \log(c_1) + \beta \log(c_2) + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

3. First-order conditions for c_1, c_2:

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta \frac{1}{c_2} - \lambda \frac{1}{1+r} = 0 \tag{9}$$

3. Euler equation: (8) + (9)

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2} \tag{10}$$

4. Optimal consumption: (7) + (10)

$$c_1 = \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r} \right] \tag{11}$$

$$c_2 = \frac{\beta}{1+\beta}\left[(1+r)y_1 + y_2 \right] \tag{12}$$

Consumption response to changes in r

Log preferences and period-2 income

How does c_1 respond to changes in r?

- Eq. (11) clearly decreasing in r
- Previously c_1 did not respond at all.

  Now: $r \uparrow \implies$ PV of income $\downarrow \implies c_1 \downarrow$
- Often referred to as wealth effect, but terminology varies

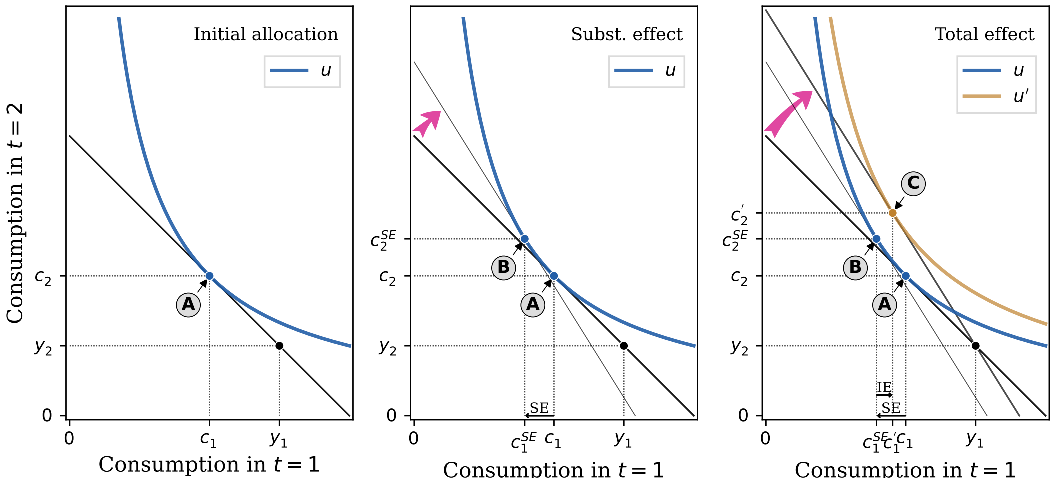


Figure 2: Income and substitution effects of an increase in r for a lender with log preferences and income in both periods

Summary: Consumption response to changes in r

Log preferences and period-2 income

Summary: Decomposition for lender as r increases

Decomposition	$\partial c_1 / \partial r$	
Substitution effect	< 0	
Income effect	> 0	
Wealth effect	≤ 0	Depends on timing of income
Total effect	?	

Table 1: Decomposition of change in lender's period-1 consumption following an increase in r

What about borrowers? — See exercises

What about decrease in r?

Consumption growth and the EIS

What determines magnitude of substitution effect?

- Or equivalently: what determines changes in consumption growth c_2/c_1?
- Previous graphs suggest link to curvature of indifference curves
- We want to formalise this *willingness* to shift consumption as r changes
  - Characterised by elasticity of intertemporal substitution (EIS)
  - Log preferences restricted to EIS = 1, so we study general CRRA preferences

Household problem with CRRA preferences

- Household solves:

$$\max_{c_1,\, c_2,\, a_2} \quad \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_1 + a_2 = a_1 + y_1$$

$$c_2 = (1+r)a_2 + y_2$$

New: CRRA utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$

Recall from unit 1: CRRA preferences

- Most frequently used preference class in macroeconomics
- Special case: logarithmic preferences
- Utility function given by

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases}$$

Note: in economics log *almost always denotes the natural logarithm!*

- Parameter γ is called the coefficient of relative risk aversion (RRA)

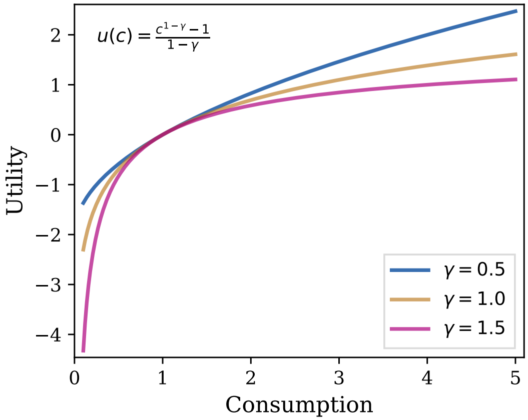


Figure 3: CRRA utility for different values of the relative risk aversion parameter γ.

Solving the problem: CRRA preferences

1. Lifetime budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{13}$$

2. Lagrangian:

$$\mathcal{L} = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma} + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

3. First-order conditions for c_1, c_2:

$$\frac{\partial \mathcal{L}}{\partial c_1} = c_1^{-\gamma} - \lambda = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta c_2^{-\gamma} - \lambda \frac{1}{1+r} = 0 \tag{15}$$

3. Euler equation: (14) + (15)

$$c_1^{-\gamma} = \beta(1+r)c_2^{-\gamma} \tag{16}$$

4. Optimal consumption growth from (16)

$$\frac{c_2}{c_1} = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} \tag{17}$$

Don't need to fully solve for optimal c_1, c_2 to say something about the SE

Goal: Approximate consumption growth in (17)

Steps 1–3

1. For small x, we have $\log(1+x) \approx x$. Apply to consumption ratio:

$$\log(c_2/c_1) = \log\left(1 + \frac{c_2 - c_1}{c_1}\right) \approx \frac{c_2 - c_1}{c_1} \tag{18}$$

2. Take logs in (17):

$$\begin{aligned}\frac{c_2 - c_1}{c_1} &\approx \log(c_2/c_1) = \log\left(\left[\beta(1+r)\right]^{\frac{1}{\gamma}}\right) = \frac{1}{\gamma}\Big[\log(1+r) + \log(\beta)\Big] \\ &\approx \frac{1}{\gamma}\left[r + \log(\beta)\right] \end{aligned} \tag{19}$$

3. Define rate of time preference ρ such that $\beta \equiv \frac{1}{1+\rho}$

$$\log(\beta) = \log\left(\frac{1}{1+\rho}\right) = \log(1) - \log(1+\rho) \approx -\rho \tag{20}$$

Step 4

4. Plug (20) into (19) to get approximate consumption growth rate:

$$\frac{c_2 - c_1}{c_1} \approx \frac{1}{\gamma}(r - \rho) \tag{21}$$

Interpretation?

Consumption growth depends on sign of $r - \rho$

- $r > \rho$: Market return higher than time preference rate $\implies$ HH shifts consumption to period 2
- $r = \rho$: Market and HH discount future at same rate, $c_2 = c_1$
- $r < \rho$: HH discounts future more heavily $\implies$ shifts consumption to period 1

$\frac{1}{\gamma}$ governs how strongly household responds to gap in $r - \rho$

Elasticity of intertemporal substitution (EIS)

What exactly is this $\frac{1}{\gamma}$?

- We show that this is the elasticity of c_2/c_1 with respect to $(1+r)$

Recall from microeconomics:

Definition (Elasticity)

The elasticity of y with respect to x is defined as

$$Elasticity = \frac{dy/y}{dx/x} = \frac{dy}{dx}\frac{x}{y} = \frac{d\log y}{d\log x}$$

Interpretation

Unit-free measure that links relative changes in y to relative changes in x.

$$\underbrace{dy/y}_{\%\text{ change in } y} = Elasticity \times \underbrace{dx/x}_{\%\text{ change in } x}$$

Elasticity of intertemporal substitution (EIS)

In the context of our consumption-savings model:

Definition (EIS)

The elasticity of intertemporal substitution (EIS) is

$$EIS = \frac{d\log\left(c_2/c_1\right)}{d\log(1+r)} \tag{22}$$

Find expression for elasticity (22):

1. Take logs of (17):

$$\log(c_2/c_1) = \frac{1}{\gamma}\log\beta + \frac{1}{\gamma}\log(1+r)$$

2. Take derivative w.r.t. $\log(1+r)$

$$EIS = \frac{d\log\left(c_2/c_1\right)}{d\log(1+r)} = \frac{1}{\gamma}$$

EIS is a constant $\implies$ isolastic preferences!

Elasticity of intertemporal substitution (EIS)

Summary of findings

- For CRRA preferences, $EIS = \frac{1}{RRA} = \frac{1}{\gamma}$
- EIS does not depend on specific values of c_2/c_1
- EIS governs how c_2/c_1 responds to changes in r:
  - Low EIS: Consumption is inelastic (the substitution effect is small)
    Even large changes in r move c_2/c_1 only by small mount.
  - EIS = 1: Log preferences
  - High EIS: Consumption is elastic (the substitution effect is large)
    Small changes in r can move c_2/c_1 a lot!

Important: Our findings assume an interior solution — constrained HH might not respond at all to changes in r.

Elasticity of intertemporal substitution (EIS)

Graphical illustration of small (left) vs. large (right) EIS

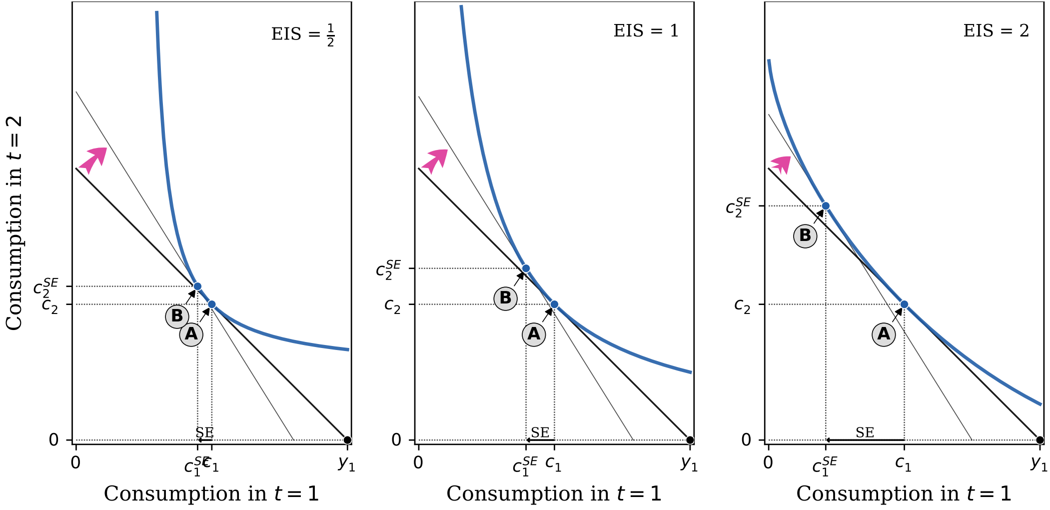


Figure 4: Substitution effect of an increase in r for different EIS values.

Life cycle model with many periods

Life cycle model with two periods

Two-period model as stylised life cycle

Period 1: Household receives income, represents ≈ 45 years of working life

Period 2: Retirement, household lives off savings from period 1

Example: Figure 5 with $\beta = 1$, $r = 0$

- Income received only in the first period
- Consumption is perfectly smoothed across both periods
- Saving equals dissaving

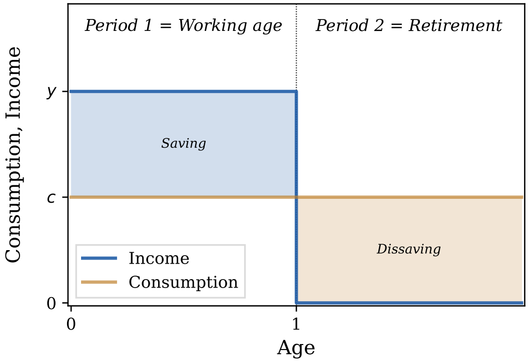


Figure 5: Stylised two-period life cycle model

Life cycle model with many periods

Natural extension to many periods:

- Life span of $T = 60$
- Age $t = 0, 1, \ldots, T-1$
- Working life of $N = 45$ periods

Example: Figure 6 with $\beta = 1$, $r = 0$

- Constant income while working, no income in retirement:

$$y_t = \begin{cases} y & \text{if } t < N \\ 0 & \text{if } t \geq N \end{cases}$$

- Consumption is perfectly smoothed across all periods

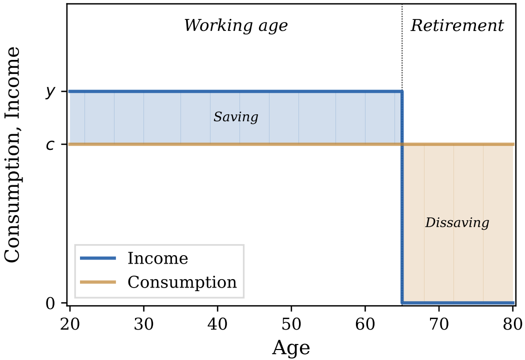


Figure 6: Stylised 60-period life cycle model

Life cycle model with CRRA preferences

Maximisation problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t) \tag{23}$$

$$\text{s.t.} \quad c_t + a_{t+1} = (1+r)a_t + y_t \qquad \forall\ t \tag{24}$$

$$a_T \geq 0, \quad a_0 \ \text{ given} \tag{25}$$

- CRRA preferences $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$
- In each period, household chooses c_t and a_{t+1} for all $t = 0, 1, \ldots, T-1$
- Receives per-period income y_t
- Household cannot die in debt: $a_T \geq 0$

Solving the problem

Life cycle model with many periods

- Lifetime budget constraint can be derived by repeated substitution

$$\underbrace{\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t}}_{\text{PV of cons.}} = \underbrace{(1+r)a_0}_{\text{Init. wealth}} + \underbrace{\sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t}}_{\text{PV of income}} \tag{26}$$

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[(1+r)a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} - \sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} \right] \tag{27}$$

- First-order condition for c_t in <u>any</u> period t:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \frac{\lambda}{(1+r)^t} = 0 \tag{28}$$

Solving the problem: Euler equation

- We need to eliminate λ in (28). Use FOC for c_{t+1}:

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \beta^{t+1} u'(c_{t+1}) - \frac{\lambda}{(1+r)^{t+1}} = 0 \tag{29}$$

- Solve for λ in (28) and (29), equate expressions:

$$\beta^t (1+r)^t u'(c_t) = \beta^{t+1} (1+r)^{t+1} u'(c_{t+1})$$

- Cancel common terms to obtain Euler equation:

$$u'(c_t) = \beta(1+r) u'(c_{t+1})$$

For CRRA preferences:

$$c_t^{-\gamma} = \beta(1+r) c_{t+1}^{-\gamma} \tag{30}$$

Example:
Model with constant income and retirement

Example: Model with constant income and retirement

Let's solve the example shown in Figure 6:

- HH lives for $T = 60$ periods, working life of $N = 45$ periods
- No initial assets, $a_0 = 0$
- Assume $\beta = 1$, $r = 0$
- Income constant while working, no income in retirement:

$$y_t = \begin{cases} y & \text{if } t < N \\ 0 & \text{if } t \geq N \end{cases}$$

Solving the problem

- From Euler equation (30):

$$c_t^{-\gamma} = \beta(1+r)c_{t+1}^{-\gamma} \implies c_t^{-\gamma} = c_{t+1}^{-\gamma} \implies c_t = c_{t+1}$$

Consumption is constant, $c_t = c$ for all t

Solving the problem

Model with constant income and retirement

Find optional c from lifetime budget constraint:

1. PV of lifetime consumption (l.h.s. of (26)):

$$\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{T-1} c = Tc \tag{31}$$

2. PV of lifetime income (r.h.s. of (26)):

$$(1+r)a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} = \sum_{t=0}^{N-1} y = Ny \tag{32}$$

3. Use LTBC, solve for c:

$$Tc = Ny \implies c = \frac{N}{T}y \tag{33}$$

While working, each period the household

- consumes fraction N/T of income
- saves fraction $(1 - N/T)$ for retirement

Lifecycle profiles of income, consumption and assets

Model with constant income and retirement

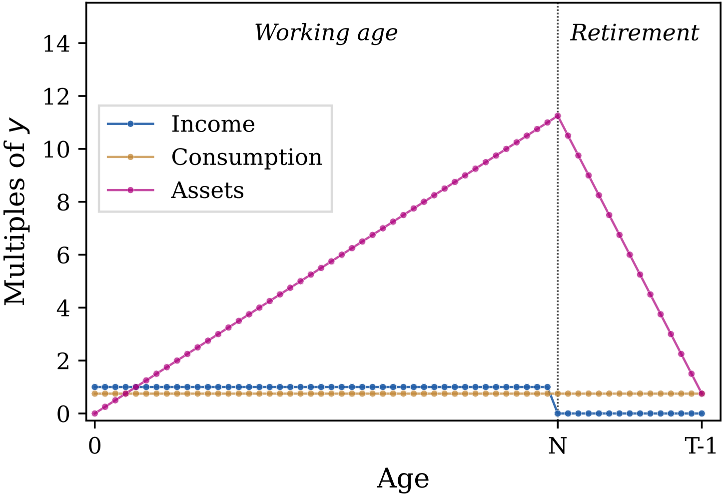


Figure 7: Life cycle profiles of income, consumption and assets for model with log preferences, $r = 0$ and $\beta = 1$. Dots indicate choices at each age.

Example:
Model with log preferences and discounting

Example: Model with log preferences and discounting

Small extension to previous example:

- HH discounts future with $\beta < 1$
- Log preferences: $\gamma = 1$
- Remaining parameters unchanged

Solving the problem

- From Euler equation (30):

$$c_t^{-\gamma} = \beta(1+r)c_{t+1}^{-\gamma} \implies c_t^{-1} = \beta c_{t+1}^{-1} \implies c_{t+1} = \beta c_t$$

- Expression c_t as function of c_0:

$$\begin{aligned} c_1 &= \beta c_0 \\ c_2 &= \beta c_1 = \beta^2 c_0 \\ &\vdots \\ c_t &= \beta^t c_0 \end{aligned}$$

Consumption no longer constant!

Solving the problem

Model with log preferences and discounting

Find optional c from lifetime budget constraint:

1. PV of lifetime consumption (l.h.s. of (26)):

$$\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{T-1} \beta^t c_0 = c_0 \sum_{t=0}^{T-1} \beta^t = c_0 \left[1 + \beta + \beta^2 + \cdots + \beta^{T-1}\right] = c_0 \frac{1-\beta^T}{1-\beta}$$

2. PV of lifetime income (r.h.s. of (26)) — unchanged from earlier:

$$(1+r)a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} = \sum_{t=0}^{N-1} y = Ny$$

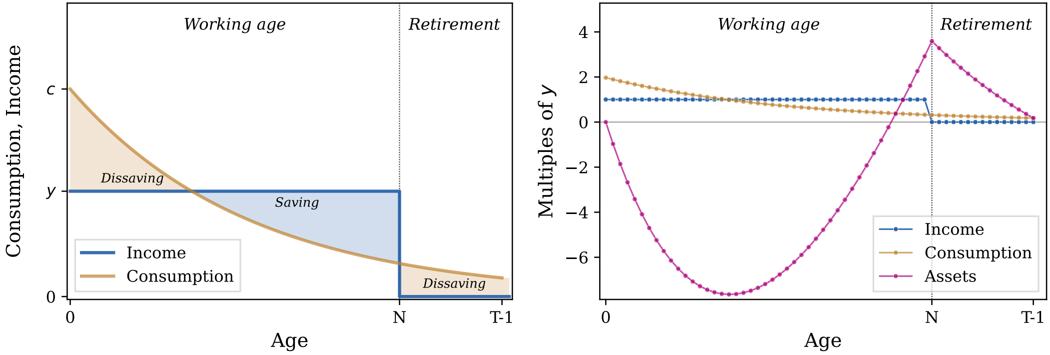
3. Use LTBC, solve for c:

$$c_0 \frac{1-\beta^T}{1-\beta} = Ny$$

$$\implies \quad c_0 = \frac{1-\beta}{1-\beta^T} Ny = \frac{1}{1+\beta+\beta^2+\cdots+\beta^{T-1}} Ny$$

Now $c_0 > \frac{N}{T} y$ as HH is more impatient!

Numerical example with $\beta = 0.96$: $c_0 = 1.97 > y = 1$



(a) Saving/dissaving over the life cycle

(b) Profiles for income, consumption and assets

Figure 8: Life cycle profiles of income, consumption and assets for model with log preferences, $r = 0$ and $\beta = 0.96$

Example:
Consumption growth in general CRRA model

Consumption growth and EIS

- Generalising the model to $r \neq 0$, $\gamma \neq 1$, etc. makes solution much more tedious
- However, we can say something about consumption growth just from Euler equation in (30):

$$\frac{c_{t+1}}{c_t} = \left[\beta(1+r)\right]^{\frac{1}{\gamma}}$$

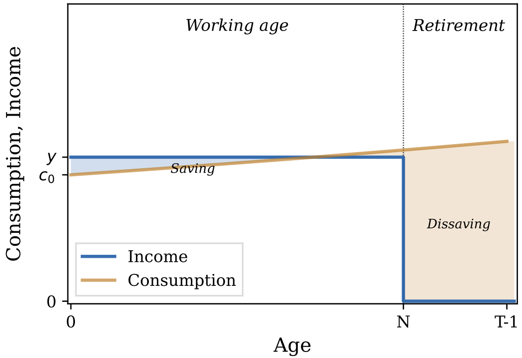
Example: low vs. high EIS

- Let $\beta = 0.96$, $r = 0.05 \implies \beta(1+r) > 1 \implies \frac{c_{t+1}}{c_t} > 1$
- Household will want to save, consume later in life!
- Two EIS scenarios:

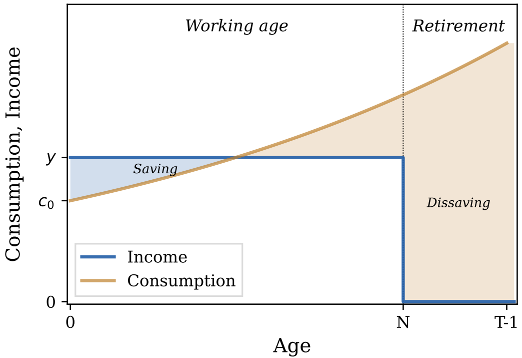
  EIS = $\frac{1}{2}$: Low consumption growth
  EIS = 2: High consumption growth

Consumption/savings over the life cycle

Low vs. high EIS



(a) EIS = $\frac{1}{2}$



(b) EIS = 2

Figure 9: Income and consumption profiles for different EIS values with $\beta = 0.96$ and $r = 0.05$.

Asset profiles over the life cycle

Low vs. high EIS

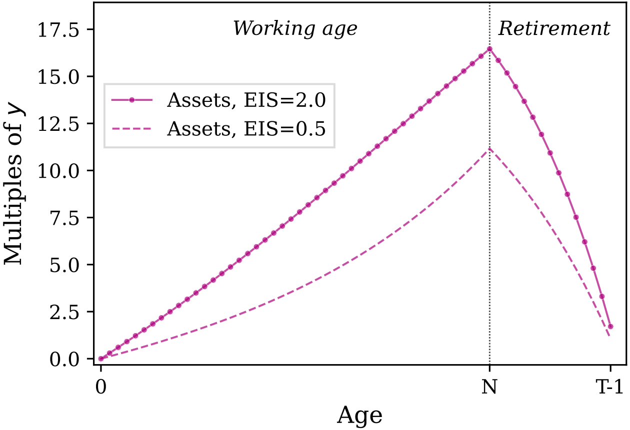


Figure 10: Life cycle profiles for assets for different EIS values with $\beta = 0.96$ and $r = 0.05$.

Life cycle model with earnings growth

- In the data, most people have growing earnings trajectories
  - Take income profile from Cocco, Gomes, and Maenhout (2005)
- Set $(1 + r) = \beta^{-1} = 1.04$
- HH borrows against rising future income!

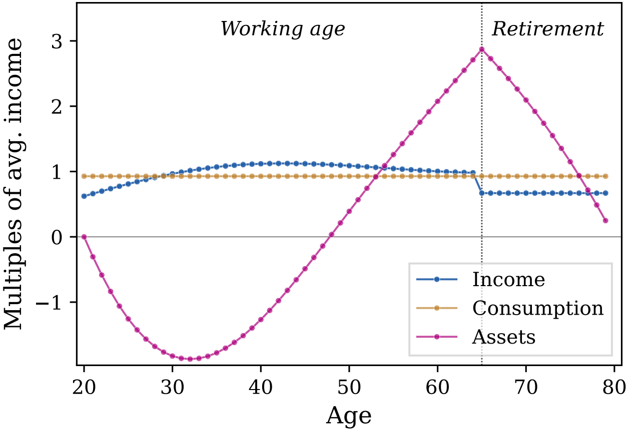


Figure 11: Life cycle profiles for income, consumption and assets.

Life cycle profiles in the data

Model predictions vs. data

Predictions from our (simple) life cycle model with borrowing:

1. Households smooth consumption
   - Consumption disconnected from income in that particular period
   - Perfect consumption smoothing if $r = \rho$
2. Asset position adjusts to bridge gap between consumption and income:
   - Rising income profile $\implies$ borrowing early in life
   - Assets approach zero as household approaches end of life

Do these predictions hold in the data?

Data: Consumption vs. income in the UK

Household income and consumption by age and education

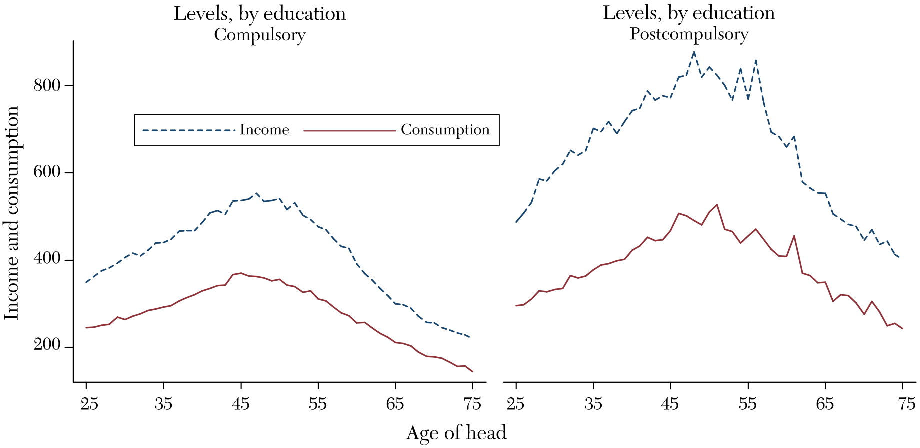


Figure 12: Average income and (nondurable) consumption by education in £/week. Source: Attanasio and Weber (2010, Figure 1), based on UK Family Expenditure Survey 1978–2007

Data: Consumption vs. income in the UK

Household income and consumption by age, education and cohort.

- Older cohorts are poorer, controlling for this flattens profiles!

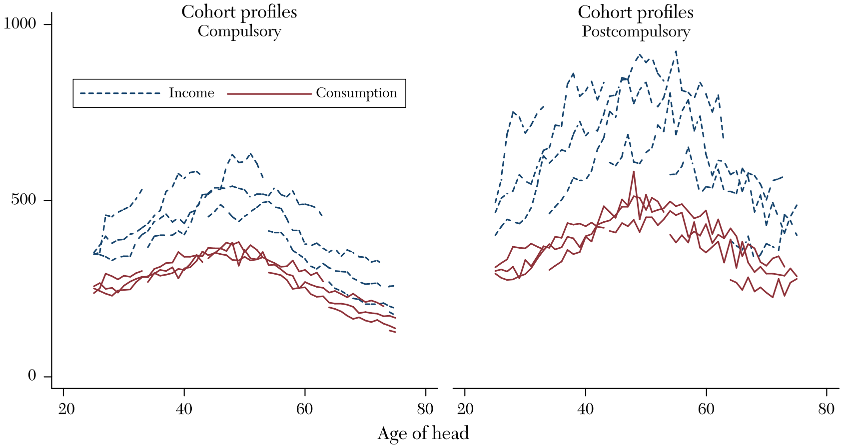


Figure 13: Average income and (nondurable) consumption by cohort and education in £/week. Source: Attanasio and Weber (2010, Figure 1), based on UK Family Expenditure Survey 1978–2007

Per capita household income and consumption by age, education and cohort

- Controlling for household size flattens profiles even more!

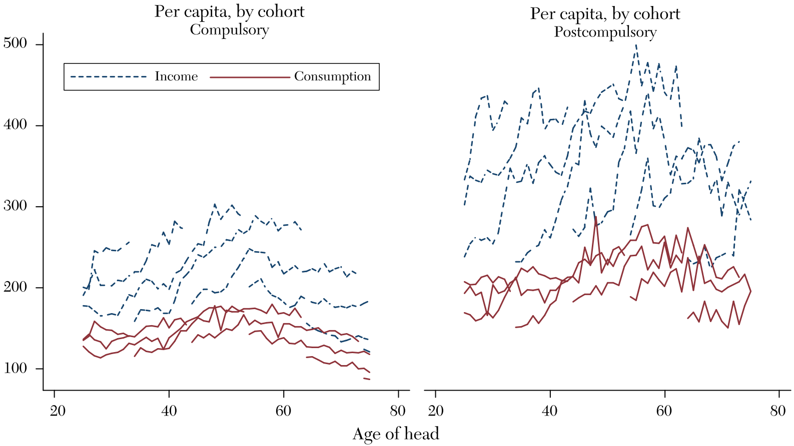
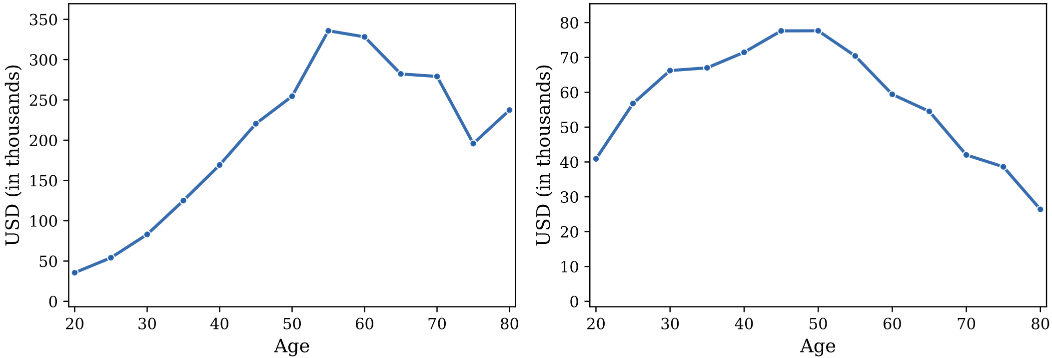


Figure 14: Average per capita income and (nondurable) consumption by cohort and education in £/week. Source: Attanasio and Weber (2010, Figure 1), based on UK Family Expenditure Survey 1978–2007

Data: Net worth in the US

Some evidence for consumption smoothing, but asset profile looks nothing like model prediction!



(a) Net worth

(b) Labour income

Figure 15: Median net worth and gross household labour income (incl. retirement benefits) in thousands of 2009 USD. Medians are computed within 5-year age bins. Data source: SCF 1998–2007

Data: Net worth in the US

- With rising earnings profile as in Figure 15b, model predicts borrowing in early life
- Median household has positive net worth at all ages (including housing and mortgages)
- High levels of asset holdings until old age:
  - Bequest motives?
  - Insurance against health shocks and long-term care needs?
  - Net worth mostly due to primary residence? — HH do not want to or cannot downsize

Main takeaways from this unit

Main takeaways

Models / theory

We introduced the following concepts:

1. Decomposition of consumption responses to changes in r:
   - Substitution effect (SE) due to change in relative price
   - Income effect (IE) for lenders/borrowers
   - Wealth effect due to change in present value of future income
2. Elasticity of intertemporal substitution: quantifies magnitude of SE
3. Life cycle model: extension of two-period models to working life and retirement phases with many periods.

Life cycle profiles in the data

1. Some support for consumption smoothing
2. Asset profiles look very different from predictions of our (simple) life cycle model

References

Attanasio, Orazio P. and Guglielmo Weber (2010). "Consumption and Saving: Models of Intertemporal Allocation and Their Implications for Public Policy". In: **Journal of Economic Literature** 48.3, pp. 693–751.

Cocco, João F., Francisco J. Gomes, and Pascal J. Maenhout (2005). "Consumption and Portfolio Choice over the Life Cycle". In: **Review of Financial Studies** 18.2, pp. 491–533.

Unit 3: Uncertainty – Complete Markets

Advanced Macroeconomics (ECON4040) – Part 2

Richard Foltyn

March 3, 2023

Outline for today

Uncertainty

Uncertainty in economics

So far, all our models were deterministic: households knew all realisations of income and returns in advance.

Give examples of economically relevant uncertainty!

- Labour earnings
- Unemployment
- Investment returns (e.g., stock returns)
- Survival
- Health state
- Divorce / separation

Uncertainty in economic models

Deterministic household problem

$$\max_{c_1, c_2, a_2} \quad u(c_1) + \beta u(c_2)$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = (1 + r)a_2 + y_2$$

y_2 — Deterministic income

r — Deterministic asset return

Stochastic household problem

$$\max_{c_1, c_2, a_2} \quad u(c_1) + \beta\, \mathbf{E}\left[\, u(c_2) \,\right]$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = (1 + r_2)a_2 + y_2$$

y_2 — Uncertain income

r_2 — Uncertain asset return

With incomplete markets, uncertainty creates ex post heterogeneity:

- some individuals had good, others bad draws
- even true if everyone was identical ex ante

How do we model uncertainty?

- Terminology
  - In this course: uncertainty and risk are used as synonyms
  - Something uncertain is stochastic or random
  - Something certain is often called deterministic
- Formally modelled as a random variable
  - Well-defined framework to quantify uncertain events
  - We ignore technical details, focus on simplest form of uncertainty
- Agents are perfectly informed about the true process generating uncertainty (rational expectations)
  - Parameters governing this process influence household choices

Common distributions used in macroeconomics

Continuous random variables

- Normal (Gaussian): $X \sim \mathcal{N}(\mu, \sigma^2)$
  - Used for: asset returns
  - Expected value of function $f(X)$:

$$\mathbf{E}\left[\, f(X) \,\right] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(x) e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

- Log-normal: $\log X \sim \mathcal{N}(\mu, \sigma^2)$
  - Used for: labour earnings, asset returns
  - Expected value of function $f(X)$:

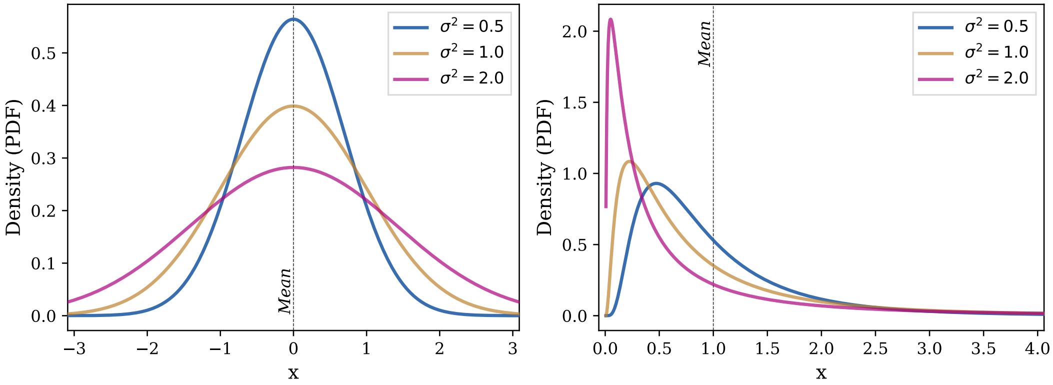
$$\mathbf{E}\left[\, f(X) \,\right] = \frac{1}{\sqrt{2\pi}\sigma} \int_{0}^{\infty} \frac{f(x)}{x} e^{-\frac{1}{2}\left(\frac{\log(x)-\mu}{\sigma}\right)^2} dx$$

- Pareto
  - Used for: firm productivity, top incomes

Discrete random variables

- Bernoulli
  - Outcome either 0 or 1; 1 occurs with probability π
  - User for: exogenous unemployment shocks
- Generalised Bernoulli
  - Extended to multiple outcomes

Common continuous distributions



(a) Normal **(b)** Log-normal

Figure 1: Probability density functions for normal and log-normal distributions

Income as a **discrete** random variable

- We focus on labour income as source of uncertainty
- Assume income y_{t+1} is a random variable with two possible realisations:
  - y_b "bad"
  - y_g "good"

$$y_{t+1} = \begin{cases} y_b & \text{with probability } \pi \\ y_g & \text{with probability } 1-\pi \end{cases}$$

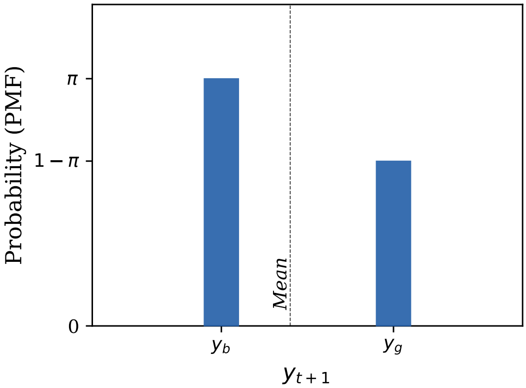


Figure 2: Discrete random variable with possible realisations (y_b, y_g)

Mean and variance

Distributions are characterised by so-called moments:

1. Mean (or expected value): 1st moment
2. Variance: 2nd (central) moment

Mean of discrete random variable

- Weighted sum of all possible realisations
- Weights are given by realisation probabilities $\Pr(y_{t+1} = y_i)$

$$\mathrm{E}_t y_{t+1} = y_b \cdot \Pr(y_{t+1} = y_b) + y_g \cdot \Pr\left(y_{t+1} = y_g\right) = y_b \pi + y_g (1 - \pi)$$

Mean and variance

Variance of discrete random variable

- Measure of dispersion around the mean
  Standard deviation = $\sqrt{\text{Variance}}$
- Defined as $\text{Var}\left(y_{t+1} \right) = \mathbf{E}_t y_{t+1}^2 - (\mathbf{E}_t y_{t+1})^2 = \mathbf{E}_t \left[(y_{t+1} - \mathbf{E}_t y_{t+1})^2 \right]$
- For our two-state income process:

$$\text{Var}\left(y_{t+1} \right) = \underbrace{y_b^2 \pi + y_g^2 (1-\pi)}_{\mathbf{E}_t y_{t+1}^2} - \underbrace{\left[y_b \pi + y_g (1-\pi) \right]^2}_{(\mathbf{E}_t y_{t+1})^2}$$

$$\vdots$$

$$= \pi(1-\pi) \left[y_b - y_g \right]^2$$

Intuition?

- Variance increasing in distance $|y_g - y_b|$ (outcomes are more dispersed)
- Variance maximised at $\pi = \frac{1}{2}$ (both outcomes equally likely)

Example: symmetric income risk

Symmetric income risk

Income given by

$$y_{t+1} = \begin{cases} y - \epsilon & \text{with prob. } \frac{1}{2} \\ y + \epsilon & \text{with prob. } \frac{1}{2} \end{cases}$$

for some fixed ϵ with $0 < \epsilon < y$.

Moments:

$$\mathbf{E}_t y_{t+1} = \frac{1}{2}(y - \epsilon) + \frac{1}{2}(y + \epsilon) = y$$

$$\text{Var}\,(\,y_{t+1}\,) = \epsilon^2$$

Mean-preserving spread

Income given by

$$y_{t+1} = \begin{cases} y - 2\epsilon & \text{with prob. } \frac{1}{2} \\ y + 2\epsilon & \text{with prob. } \frac{1}{2} \end{cases}$$

where ϵ is unchanged from before.

Moments:

$$\mathbf{E}_t y_{t+1} = y$$

$$\text{Var}\,(\,y_{t+1}\,) = 4\epsilon^2$$

Mean-preserving spread

Mean-preserving spreads leaves mean unchanged, but quadruples variance (in this example).

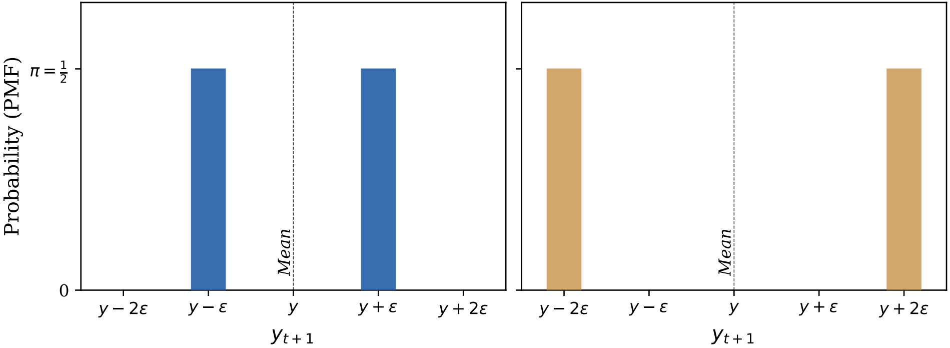


Figure 3: Mean-preserving spread from state space $(y-\epsilon, y+\epsilon)$ to $(y-2\epsilon, y+2\epsilon)$

Risk aversion

Recall from unit 1: CRRA preferences

- Utility function given by

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases}$$

- Parameter γ is called the coefficient of relative risk aversion (RRA)
- Unit 2: We showed that EIS = $\frac{1}{\gamma}$
- As name implies, RRA is also related to risk aversion
  - γ = Arrow-Pratt coefficient of relative risk aversion.
- With CRRA, two very different concepts are mapped into single parameter!

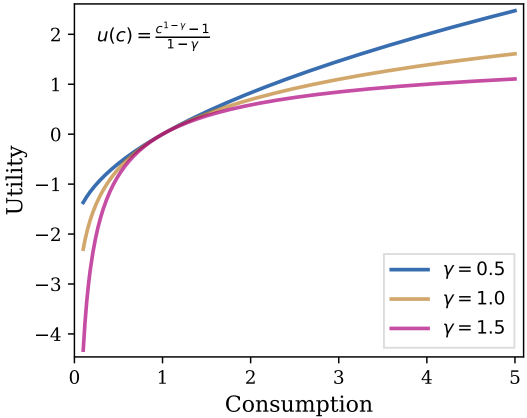


Figure 4: CRRA utility for different values of the relative risk aversion parameter γ

Quantifying risk aversion

- Magnitude of RRA parameter: higher $\gamma \implies$ more risk averse
- Certainty equivalent: higher CE $\implies$ more risk averse
- Risk premium: higher risk premium $\implies$ more risk averse

Example:

- Static setting with stochastic consumption (gamble):

$$c = \begin{cases} c_b & \text{with prob. } \pi \\ c_g & \text{with prob. } 1 - \pi \end{cases}$$

- For illustration, let $\pi = \frac{1}{2}$
- CRRA utility function $u(c)$
- Expected utility:

$$\mathbf{E}u(c) = \frac{1}{2}u(c_b) + \frac{1}{2}u(c_g)$$

Certainty equivalent and risk premium

Certainty equivalent

- Suppose individual could avoid gamble and get certain outcome CE instead

What is the lowest acceptable certain amount?

- CE must satisfy

$$u(CE) = \mathbf{E}u(c)$$

- For risk-averse individual with strictly concave $u(\bullet)$:

$$u(CE) = \underbrace{\mathbf{E}u(c) < u(\mathbf{E}c)}_{\text{Jensen's inequality}} \implies CE < \mathbf{E}c$$

Risk premium

- Difference between expected outcome and CE: $p = \mathbf{E}c - CE$

Intuition?

- Risk-averse individual dislikes gambles, accepts lower certain amount
- Risk-averse individual is willing to forfeit p in expectation

Certainty equivalent and risk premium

Graphical illustration of previous example:

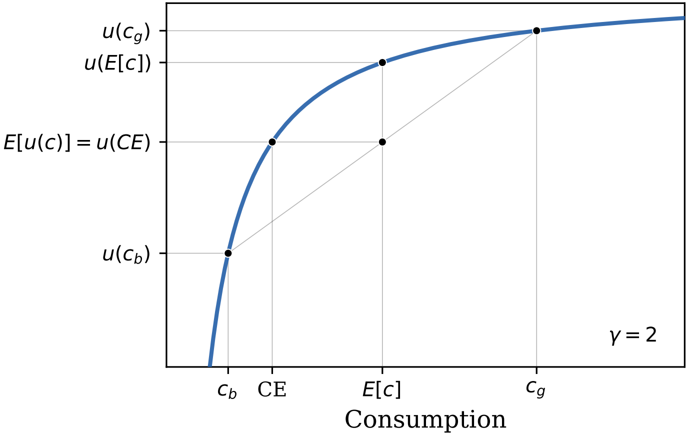


Figure 5: Certainty equivalent for individual with relative risk aversion $\gamma = 2$.

Certainty equivalent and risk premium

With CRRA, RRA parameter γ affects CE and risk premium!

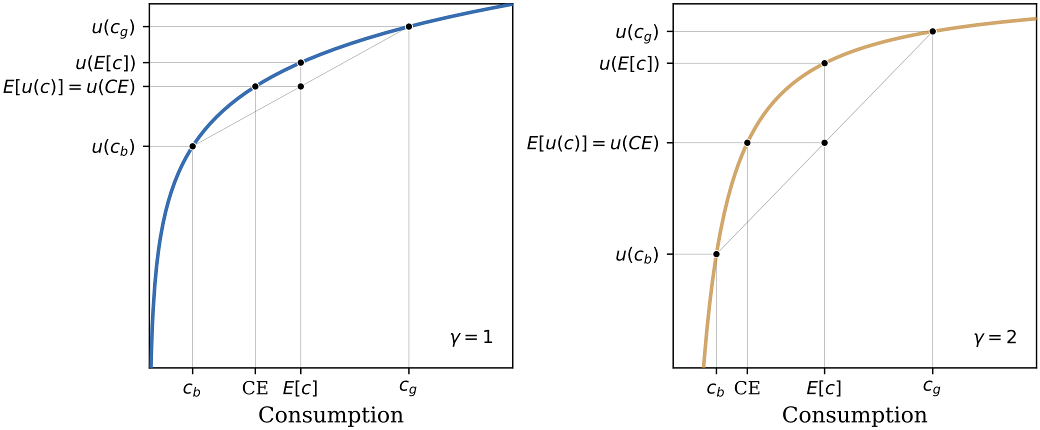


Figure 6: Certainty equivalent for RRA = 1 (left) and RRA = 2 (right)

Complete markets:
Decentralised economy

Complete markets: environment

- Simplest setup: two periods, two possible states in period 2
- Household income in $t = 2$ depends on s_2: good or bad realisation

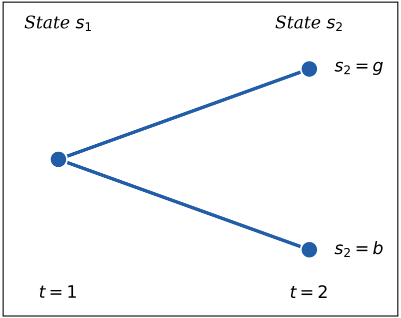


Figure 7: Event tree for two periods with uncertainty about state s_2 in the second period

- In $t = 1$, households trade contingent bonds labelled b and g:

$$\text{payoff}_b(s_2) = \begin{cases} 1 & \text{if } s_2 = b \\ 0 & \text{if } s_2 = g \end{cases}$$

$$\text{payoff}_g(s_2) = \begin{cases} 0 & \text{if } s_2 = b \\ 1 & \text{if } s_2 = g \end{cases}$$

- Each bond delivers one unit of consumption in one particular state
- Bond prices: q_b, q_g
- Such bonds are called Arrow securities

Household maximises expected utility:

$$\max_{c_1, c_{2b}, c_{2g}, a_b, a_g} \quad u(c_1) + \beta \underbrace{\left[\pi u(c_{2b}) + (1-\pi)u(c_{2g})\right]}_{\equiv \mathbf{E}u(c_2)} \tag{1}$$

$$\text{s.t.} \quad c_1 + q_b a_b + q_g a_g = y_1 \tag{2}$$

$$c_{2b} = a_b + y_{2b} \tag{3}$$

$$c_{2g} = a_g + y_{2g} \tag{4}$$

a_b: Number of Arrow bonds purchased for state b at price q_b

a_g: Number of Arrow bonds purchased for state g at price q_g

y_2: Period 2 income

$$y_2 = \begin{cases} y_{2b} & \text{with prob. } \pi \\ y_{2g} & \text{with prob. } 1-\pi \end{cases} \tag{5}$$

Solving the problem: lifetime budget constraint

Complete markets: decentralised economy

As usual, insert budget constraints (3), (4) into (2):

$$\underbrace{c_1 + q_b c_{2b} + q_g c_{2g}}_{\text{Value of LT cons.}} = \underbrace{y_1 + q_b y_{2b} + q_g y_{2g}}_{\text{Value of LT inc.}} \tag{6}$$

Alternative interpretation with complete markets:

- Income in period 2:
  - HH sells y_{2b} Arrow bonds b for unit price q_b
  - HH sells y_{2g} Arrow bonds g for unit price q_g
- Consumption in period 2:
  - HH purchases c_{2b} Arrow bonds b for unit price q_b
  - HH purchases c_{2g} Arrow bonds g for unit price q_g
- Period 1 income/consumption: price normalised to 1

HH sells entire lifetime income, purchases entire lifetime consumption in $t = 1$.

1. Lagrangian:

$$\mathcal{L} = u(c_1) + \beta\Big[\pi u(c_{2b}) + (1-\pi)u(c_{2g})\Big] + \lambda\Big[y_1 + q_b y_{2b} + q_g y_{2g} - c_1 - q_b c_{2b} - q_g c_{2g}\Big]$$

2. First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2b}} = \beta\pi u'(c_{2b}) - \lambda q_b = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2g}} = \beta(1-\pi)u'(c_{2g}) - \lambda q_g = 0 \tag{9}$$

3. EE for Arrow bond b: (7) + (8)

$$u'(c_1)q_b = \beta\pi u'(c_{2b}) \tag{10}$$

4. EE for Arrow bond g: (7) + (9)

$$u'(c_1)q_g = \beta(1-\pi)u'(c_{2g}) \tag{11}$$

General equilibrium

Solving for prices

Complete markets: decentralised economy

Two possible solution methods

1. Find optimal consumption rules, ensure market clearing
   Which markets are operational in this economy?
   1. Market for consumption in period 1
   2. Market for consumption in period 2, bad state
   3. Market for consumption in period 2, good state

   Can be very tedious, even with log preferences.
2. Use FOCs to determine equilibrium prices — we use this method!

Assumptions

- Two states in $t = 2$: $s = b, g$
- Two households $i = A, B$ with income y^i_{ts} in period t, state s
- Aggregate endowments: $Y_1 = y^A_1 + y^B_1, \quad Y_{2b} = y^A_{2b} + y^B_{2b}, \quad Y_{2g} = y^A_{2g} + y^B_{2g}$

1 FOCs (7), (8) and (9) for CRRA preferences for $i = A, B$:

$$\left(c_1^i\right)^{-\gamma} = \lambda_i$$
$$\beta\pi\left(c_{2b}^i\right)^{-\gamma} = \lambda_i q_b$$
$$\beta(1-\pi)\left(c_{2g}^i\right)^{-\gamma} = \lambda_i q_g$$

2 Divide A's by B's FOCs:

$$\frac{\left(c_1^A\right)^{-\gamma}}{\left(c_1^B\right)^{-\gamma}} = \frac{\lambda_A}{\lambda_B}$$
$$\frac{\beta\pi\left(c_{2b}^A\right)^{-\gamma}}{\beta\pi\left(c_{2b}^B\right)^{-\gamma}} = \frac{\lambda_A q_b}{\lambda_B q_b}$$
$$\frac{\beta(1-\pi)\left(c_{2g}^A\right)^{-\gamma}}{\beta(1-\pi)\left(c_{2g}^B\right)^{-\gamma}} = \frac{\lambda_A q_g}{\lambda_B q_g}$$

3 Cancel common terms:

$$\left(\frac{c_1^A}{c_1^B}\right)^{-\gamma} = \left(\frac{c_{2b}^A}{c_{2b}^B}\right)^{-\gamma} = \left(\frac{c_{2g}^A}{c_{2g}^B}\right)^{-\gamma} = \frac{\lambda_A}{\lambda_B}$$

Summary of findings

- Ratio of A's to B's consumption is

$$\frac{c_1^A}{c_1^B} = \frac{c_{2b}^A}{c_{2b}^B} = \frac{c_{2g}^A}{c_{2g}^B} = \left(\frac{\lambda_A}{\lambda_B}\right)^{-\frac{1}{\gamma}} \tag{12}$$

  in all periods and all states!
- Implies that A's consumption is some constant fraction α of aggregate output (analogous for B):

$$c_1^A = \alpha \underbrace{\left(y_1^A + y_1^B\right)}_{\equiv Y_1}, \qquad c_{2b}^A = \alpha \underbrace{\left(y_{2b}^A + y_{2b}^B\right)}_{\equiv Y_{2b}}, \qquad c_{2g}^A = \alpha \underbrace{\left(y_{2g}^A + y_{2g}^B\right)}_{\equiv Y_{2g}} \tag{13}$$

How does A's consumption depend on A's income?

- With complete markets, consumption only depends on aggregates!

We can use this insight to solve for prices. Plug (13) into Euler equations (10) and (11):

- Arrow bond b:

$$\left(c_1^A\right)^{-\gamma} q_b = \beta\pi \left(c_{2b}^A\right)^{-\gamma}$$
$$(\alpha Y_1)^{-\gamma} q_b = \beta\pi \left(\alpha Y_{2b}\right)^{-\gamma}$$
$$\implies \quad q_b = \beta\pi \left(\frac{Y_{2b}}{Y_1}\right)^{-\gamma} \qquad (14)$$

- Arrow bond g:

$$\left(c_1^A\right)^{-\gamma} q_g = \beta(1-\pi) \left(c_{2g}^A\right)^{-\gamma}$$
$$(\alpha Y_1)^{-\gamma} q_g = \beta(1-\pi) \left(\alpha Y_{2g}\right)^{-\gamma}$$
$$\implies \quad q_g = \beta(1-\pi) \left(\frac{Y_{2g}}{Y_1}\right)^{-\gamma} \qquad (15)$$

α cancels out, prices depend only on aggregates Y_1, Y_{2b} and Y_{2g}, and parameters.

Intuition? How do prices depend on aggregate income and parameters?

- Price is higher if state is more likely to occur
- Price is lower if aggregate income in that state is high

Example:
Household problem with log preferences

Solving the problem: Euler equations

Complete markets, log preferences

- Assume both HH have log preferences (we omit household index i)
- Euler equations from (10) and (11):

$$\frac{1}{c_1} q_b = \beta\pi \frac{1}{c_{2b}} \implies c_{2b} = \beta\pi \frac{1}{q_b} c_1 \tag{16}$$

$$\frac{1}{c_1} q_g = \beta(1-\pi) \frac{1}{c_{2g}} \implies c_{2g} = \beta(1-\pi) \frac{1}{q_g} c_1 \tag{17}$$

- Denote lifetime income as $\overline{y} \equiv y_1 + q_b y_{2b} + q_g y_{2g}$
- Plug (16) + (17) into LTBC (6), solve for c_1:

$$c_1 + q_b c_{2b} + q_g c_{2g} = \overline{y}$$

$$c_1 + q_b \underbrace{\beta\pi \frac{1}{q_b} c_1}_{=c_{2b}} + q_g \underbrace{\beta(1-\pi) \frac{1}{q_g} c_1}_{=c_{2g}} = \overline{y}$$

$$c_1 \left[1 + \beta\pi + \beta(1-\pi)\right] = \overline{y} \implies c_1 = \frac{1}{1+\beta} \overline{y} \tag{18}$$

Solving the problem: optimal solution

Complete markets, log preferences

Use (16), (17) and (18) to find optimal consumption in all periods/states:

$$c_1 = \frac{1}{1+\beta}\bar{y} \tag{19}$$

$$c_{2b} = \beta\pi\frac{1}{q_b}c_1 = \frac{\beta}{1+\beta}\frac{\pi}{q_b}\bar{y} \tag{20}$$

$$c_{2g} = \beta(1-\pi)\frac{1}{q_g}c_1 = \frac{\beta}{1+\beta}\frac{1-\pi}{q_g}\bar{y} \tag{21}$$

Looks almost like solution without uncertainty!

Why?

- Household insured against all idiosyncratic risk
- Irrelevant for consumption whether household turned out to be lucky ex post

Example:
Symmetric shocks & constant aggregate endowment

Example: Symmetric (negatively correlated) income risk

Complete markets, log preferences, symmetric shocks

- Continue with previous example
- Remaining object to pin down is $\overline{y}$ — need assumptions on individual income!
- Period-2 income:
  - Household A:

$$y_2^A = \begin{cases} y_{2b}^A = y_2 - \epsilon & \text{with prob. } \pi \\ y_{2g}^A = y_2 + \epsilon & \text{with prob. } 1-\pi \end{cases} \qquad \text{where } 0 < \epsilon < y_2$$

  - Household B's income realisations are flipped
- Income distribution and aggregates:

Household	Income in $t = 1$	Income in $t = 2$	
		State b (prob. π)	State g (prob. $1-\pi$)
A	y_1	$y_2 - \epsilon$	$y_2 + \epsilon$
B	y_1	$y_2 + \epsilon$	$y_2 - \epsilon$
Aggregate	$Y_1 = 2y_1$	$Y_2 = 2y_2$	$Y_2 = 2y_2$

Table 1: State-dependent income distribution

Value of lifetime income

Complete markets, log preferences, symmetric shocks

1. With log preferences and constant $Y_{2b} = Y_{2g} = Y_2$, prices (14) and (15) are

$$q_b = \beta\pi\frac{Y_1}{Y_2}$$
$$q_g = \beta(1-\pi)\frac{Y_1}{Y_2}$$

2. Lifetime income for $i = A, B$:

$$\begin{aligned}\bar{y}^i &= y_1 + q_b y_{2b}^i + q_g y_{2g}^i \\ &= y_1 + \beta\pi\frac{Y_1}{Y_2}y_{2b}^i + \beta(1-\pi)\frac{Y_1}{Y_2}y_{2g}^i \\ &= y_1 + \beta\frac{Y_1}{Y_2}\underbrace{\left[\pi y_{2b}^i + (1-\pi)y_{2g}^i\right]}_{\mathbf{E}y_{2s}^i}\end{aligned}$$

3. Assume $\pi = \frac{1}{2}$:

$$\mathbf{E}y_{2s}^A = \frac{1}{2}(y_2 - \epsilon) + \frac{1}{2}(y_2 + \epsilon) = y_2$$
$$\mathbf{E}y_{2s}^B = \frac{1}{2}(y_2 + \epsilon) + \frac{1}{2}(y_2 - \epsilon) = y_2$$

4. Lifetime income simplifies:

$$\begin{aligned}\bar{y}^i &= y_1 + \beta\frac{Y_1}{Y_2}y_2 \\ &= \frac{Y_1}{2} + \beta\frac{Y_1}{Y_2}\frac{Y_2}{2} \\ &= (1+\beta)\frac{1}{2}Y_1\end{aligned}$$

since $Y_1 = 2y_1, \quad Y_2 = 2y_2$

Optimal consumption

Complete markets, log preferences, symmetric shocks

- Optimal consumption: plug lifetime income and prices into (19), (20) and (21):

$$c_1^i = \frac{1}{1+\beta}\overline{y}^i = \frac{1}{1+\beta}(1+\beta)\frac{1}{2}Y_1 = \frac{1}{2}Y_1$$
$$c_{2b}^i = \frac{\beta}{1+\beta}\frac{\pi}{q_b}\overline{y}^i = \frac{1}{1+\beta}\frac{Y_2}{Y_1}\overline{y}^i = \frac{1}{1+\beta}\frac{Y_2}{Y_1}(1+\beta)\frac{1}{2}Y_1 = \frac{1}{2}Y_2$$
$$c_{2g}^i = \frac{\beta}{1+\beta}\frac{1-\pi}{q_g}\overline{y}^i = \frac{1}{1+\beta}\frac{Y_2}{Y_1}\overline{y}^i = \frac{1}{1+\beta}\frac{Y_2}{Y_1}(1+\beta)\frac{1}{2}Y_1 = \frac{1}{2}Y_2$$

- Households are ex ante identical $\implies$ consume exactly the same amount ex post
- Individual shock realisations do not matter (perfect insurance)

Equilibrium prices

Complete markets, log preferences, symmetric shocks

Using (14) and (15), we find equilibrium prices for Arrow bonds:

$$q_b = \beta\pi\left(\frac{Y_{2b}}{Y_1}\right)^{-\gamma} = \beta\frac{1}{2}\frac{Y_1}{Y_2}$$

$$q_g = \beta(1-\pi)\left(\frac{Y_{2g}}{Y_1}\right)^{-\gamma} = \beta\frac{1}{2}\frac{Y_1}{Y_2}$$

Because aggregate endowment and realisation probabilities are the same in both states, Arrow bond prices are identical.

What is the price of a risk-free bond in this economy?

- Create risk-free bond by purchasing one of each Arrow security
- Price of risk-free bond:

$$q = q_b + q_g = \beta\frac{1}{2}\frac{Y_1}{Y_2} + \beta\frac{1}{2}\frac{Y_1}{Y_2} = \beta\frac{Y_1}{Y_2}$$

- Risk-free interest rate:

$$(1+r) = \frac{1}{q} = \beta^{-1}\frac{Y_2}{Y_1}$$

Planner’s solution (centralised economy)

Social planner problem

Recall first fundamental theorem of welfare economics:

Definition (First welfare theorem)

Loosely speaking, a decentralised equilibrium with

- complete markets
- complete information
- perfect competition

is Pareto optimal.

- All of these criteria are satisfied in our setting
- Can solve planner's problem instead of decentralised equilibrium
- Caveat: need to know planner's Pareto weights for each household

Social planner problem

- Assume two households A and B with risky endowments
  - HH income allowed to depend on states b and g (no other restrictions imposed)
- Planner attaches Pareto weight θ_i to household i
- Planner directly allocates consumption, no savings (Arrow bonds) needed

Planner solves:

$$\max_{\left(c_1^i, c_{2b}^i, c_{2g}^i\right)_{i=A,B}} \sum_{i=A,B} \theta_i \left\{ u(c_1^i) + \beta \left[\pi u\left(c_{2b}^i\right) + (1-\pi) u\left(c_{2g}^i\right) \right] \right\} \tag{22}$$

$$\text{s.t.} \quad \sum_{i=A,B} c_1^i = Y_1 \tag{23}$$

$$\sum_{i=A,B} c_{2b}^i = Y_{2b} \tag{24}$$

$$\sum_{i=A,B} c_{2g}^i = Y_{2g} \tag{25}$$

Solving the planner's problem

- Lagrangian:

$$\mathcal{L} = \sum_{i=A,B} \theta_i \left\{ u(c_1^i) + \beta \left[\pi u\left(c_{2b}^i\right) + (1-\pi) u\left(c_{2g}^i\right) \right] \right\}$$
$$+ \lambda_1 \left[Y_1 - \sum_{i=A,B} c_1^i \right] + \lambda_b \left[Y_{2b} - \sum_{i=A,B} c_{2b}^i \right] + \lambda_g \left[Y_{2g} - \sum_{i=A,B} c_{2g}^i \right]$$

- First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_1^i} = \theta_i u'(c_1^i) - \lambda_1 = 0 \quad (26)$$
$$\frac{\partial \mathcal{L}}{\partial c_{2b}^i} = \theta_i \beta u'(c_{2b}^i) - \lambda_b = 0 \quad (27)$$
$$\frac{\partial \mathcal{L}}{\partial c_{2g}^i} = \theta_i \beta u'(c_{2g}^i) - \lambda_g = 0 \quad (28)$$

Solving the planner's problem

- Lagrange multipliers λ_1, λ_b and λ_g identical for all households:

$$\left.\begin{aligned}\theta_A u'(c_1^A) &= \lambda_1\\ \theta_B u'(c_1^B) &= \lambda_1\end{aligned}\right\} \implies \frac{u'(c_1^A)}{u'(c_1^B)} = \frac{\theta_B}{\theta_A}$$

Intuition? How does marg. utility depend on Pareto weights?

- Impose CRRA preferences:

$$\frac{\left(c_1^A\right)^{-\gamma}}{\left(c_1^B\right)^{-\gamma}} = \frac{\theta_B}{\theta_A} \implies \frac{c_1^A}{c_1^B} = \left(\frac{\theta_B}{\theta_A}\right)^{-\frac{1}{\gamma}} \tag{29}$$

- From (27) and (28):

$$\frac{c_{2b}^A}{c_{2b}^B} = \frac{c_{2g}^A}{c_{2g}^B} = \left(\frac{\theta_B}{\theta_A}\right)^{-\frac{1}{\gamma}} \tag{30}$$

- As in decentralised economy, relative consumption is constant across all periods/states!

Solving the planner's problem: equilibrium allocation

- Consumption at time $t = 1, 2$ in state $s = b, g$ for household A (analogous for B) follows from aggregate resource constraints (23), (24), (25) and optimality condition (29) or (30):

$$
\begin{aligned}
c_{ts}^A + c_{ts}^B &= Y_{ts} \\
c_{ts}^A + \underbrace{(\theta_B/\theta_A)^{\frac{1}{\gamma}} c_{ts}^A}_{=c_{ts}^B} &= Y_{ts} \\
c_{ts}^A \left[1 + (\theta_B/\theta_A)^{\frac{1}{\gamma}}\right] &= Y_{ts} \\
\implies \quad c_{ts}^A &= \frac{1}{1 + (\theta_B/\theta_A)^{\frac{1}{\gamma}}} Y_{st}
\end{aligned}
$$

- Higher relative weight θ_A/θ_B results in higher allocation to A
- What determines Pareto weights?
  - From (12) we see that $\theta_i = \lambda_i^{-1}$ where λ_i is i's Lagrange multiplier on LTBC
  - Intuition: HH with higher lifetime wealth is assigned higher weight to replicate the decentralised allocation

Main takeaways from this unit

Main takeaways

Uncertainty & risk aversion

1. Uncertainty is governed by the *variance* of income, returns, etc.
2. More risk-averse agents demand higher certainty equivalent, i.e., accept smaller *certain* amount to avoid gamble
3. More risk-averse agents demand higher risk premium
4. Risk aversion is connected to curvature of utility function
   - For CRRA preferences, curvature is governed by γ, which is the Arrow-Pratt coefficient of relative risk aversion

Complete markets

1. Allow households to perfectly insure against *idiosyncratic* risk
2. Household's allocation & welfare are *independent* of ex post shock realisations
3. Allocations are *Pareto optimal*, so decentralised equilibrium is identical to planner's solution with appropriate Pareto weights

Unit 4: Uncertainty – Incomplete Markets

Advanced Macroeconomics (ECON4040) – Part 2

Richard Foltyn

March 10, 2023

Outline for today

Complete vs. incomplete markets

Uncertainty in economic models

Two-period HH problem in deterministic vs. stochastic setting:

No uncertainty

$$\max_{c_1, c_2, a_2} \; u(c_1) + \beta u(c_2)$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = (1+r)a_2 + y_2$$

y_2 — Deterministic income

Complete markets

$$\max_{c_1, \{c_{2s}\}_s} \; u(c_1) + \beta \mathbf{E} u(c_2)$$

$$\text{s.t.} \quad c_1 + \sum_s q_s c_{2s} = y_1 + \sum_s q_s y_{2s}$$

y_2 — Uncertain income

Incomplete markets

$$\max_{c_1, c_2, a_2} \; u(c_1) + \beta \mathbf{E} u(c_2)$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = (1+r)a_2 + y_2$$

$$a_2 \geq -b$$

y_2 — Uncertain income

b — Borrowing limit

Complete vs. incomplete markets

Complete markets

1. Households can insure against all idiosyncratic risk
2. Allocations depend on ex ante lifetime wealth, not on ex post realisations
3. Consumption smoothing across time and states
4. Perfect aggregation, admits RA formulation even with uncertainty

Incomplete markets

1. Limited access to contingent assets (e.g., only risk-free bond)
2. Ex post consumption may depend on idiosyncratic shock realisations
3. Consumption smoothing across time, limited smoothing across states
4. Usually does not aggregate

Two-period problem with incomplete markets

Two-period problem with incomplete markets

Household problem for generic $u(\bullet)$

$$
\begin{aligned}
\max_{c_1, c_2, a_2} \quad & u(c_1) + \beta \mathbf{E} u(c_2) && (1) \\
\text{s.t.} \quad & c_1 + a_2 = y_1 && (2) \\
& c_2 = (1+r)a_2 + y_2 && (3) \\
& a_2 \geq -b, \quad b \equiv \frac{y_{min}}{1+r} && (4)
\end{aligned}
$$

where

- a_2 Savings in risk-free bond (not state contingent)
- y_2 Stochastic period-2 income
- y_{min} Lowest possible realisation of y_2, $y_{min} \geq 0$
- b Natural borrowing limit (HH can repay with certainty)

Transform to problem with single choice variable a_2 and derive the Euler equation:

1. Eliminate c_1, c_2:

$$\max_{a_2} \quad u(y_1 - a_2) + \beta \mathbf{E}\Big[u\big((1+r)a_2 + y_2\big) \Big]$$
$$\text{s.t.} \quad a_2 \geq -\frac{y_{min}}{1+r}$$

2. Lagrangian:

$$\mathcal{L} = u(y_1 - a_2) + \beta \mathbf{E}\Big[u\big((1+r)a_2 + y_2\big) \Big] + \lambda \left[a_2 + \frac{y_{min}}{1+r} \right]$$

3. First-order condition for a_2:

$$\frac{\partial \mathcal{L}}{\partial a_2} = -u'(y_1 - a_2) + \beta(1+r)\mathbf{E}\Big[u'\big((1+r)a_2 + y_2\big) \Big] + \lambda = 0$$

4. Euler equation (assuming $\lambda = 0$):

$$u'(\underbrace{y_1 - a_2}_{c_1}) = \beta(1+r)\mathbf{E}u'\big(\underbrace{(1+r)a_2 + y_2}_{c_2}\big) \tag{5}$$

Almost identical to deterministic case except for expectation.

Quadratic preferences

Quadratic preferences

Solving (5) is difficult. One possible simplification: quadratic utility function

Utility function

$$u(c) = \alpha c - \frac{\delta}{2}c^2 \qquad \alpha > 0\,,\ \delta > 0 \qquad (6)$$

Why?

- Linear marginal utility:

$$u'(c) = \alpha - \delta c \qquad (7)$$

  Easy to evaluate expectations!

Why not?

- Not monotonically increasing (bliss point)
- $\lim_{c\to 0} u(c) \neq -\infty$ (fails Inada condition)
- RRA increasing in c

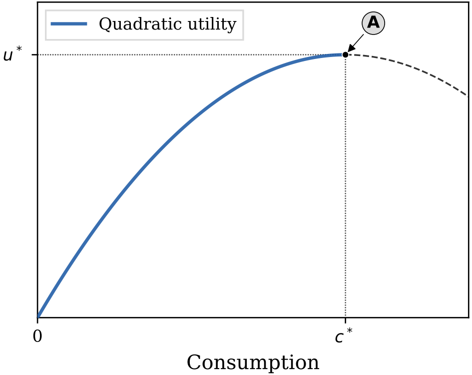


Figure 1: Quadratic utility function. (A) shows the bliss point

Quadratic preferences: certainty equivalence

- Quadratic preferences give rise to certainty equivalence
- Agent with quadratic preferences is still risk averse!

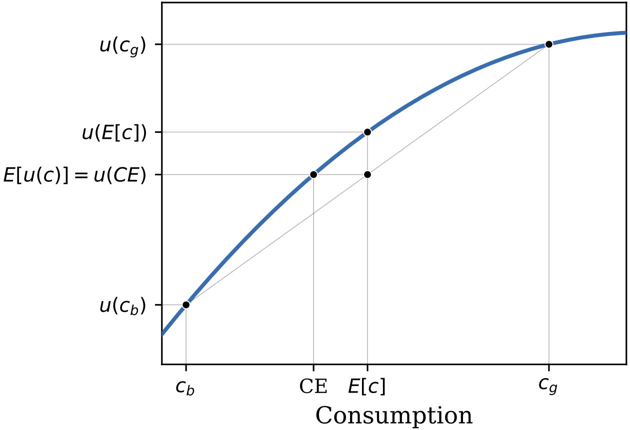


Figure 2: Certainty equivalent (CE) with quadratic preferences. The graph shows a situation in which the consumer faces a gamble with potential outcomes c_b and c_g with equal probability.

Quadratic preferences:
Deterministic model

Quadratic preferences — Deterministic model

- Solve deterministic model first, compare to stochastic variant later
- Household problem:

$$\max_{c_1, c_2, a_2} \left(\alpha c_1 - \frac{\delta}{2} c_1^2\right) + \beta\left(\alpha c_2 - \frac{\delta}{2} c_2^2\right) \tag{8}$$

$$\begin{aligned} \text{s.t.} \quad c_1 + a_2 &= y_1 \\ c_2 &= (1+r)a_2 + y_2 \\ c_1 \geq 0,\ c_2 &\geq 0 \end{aligned} \tag{9}$$

Assume that constraints (9) are satisfied.

Solving the problem

Quadratic preferences — Deterministic model

1 Lifetime budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \tag{10}$$

2 Euler equation as usual:

$$u'(c_1) = \beta(1+r)u'(c_2)$$

3 Use marg. utility from (7):

$$\alpha - \delta c_1 = \beta(1+r)\left[\alpha - \delta c_2\right]$$

4 Solve for c_2:

$$c_2 = \frac{c_1}{\beta(1+r)} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{\beta(1+r)} \tag{11}$$

Is (11) plausible?

- $\beta(1+r) = 1$: simplifies to $c_2 = c_1$

5 Substitute (11) into LTBC (10):

$$c_1 + \frac{1}{1+r}\left[\frac{c_1}{\beta(1+r)} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{\beta(1+r)}\right] = y_1 + \frac{y_2}{1+r}$$

6 Solve for c_1:

$$c_1 = \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\left[y_1 + \frac{y_2}{1+r}\right] + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \tag{12}$$

7 Savings: plug into period-1 budget constraint:

$$a_2 = y_1 - c_1 = \frac{y_1 - \beta(1+r)y_2}{1+\beta(1+r)^2} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \tag{13}$$

Solution (12) and (13) hard to understand — Look at simple cases / graphs!

Simplifications to understand results

Quadratic preferences — Deterministic model

Assume $\beta(1+r) = 1$

- c_1 from (12) simplifies to

$$c_1 = \frac{1+r}{2+r}\left[y_1 + \frac{y_2}{1+r}\right]$$

- a_2 from (13) simplifies to

$$a_2 = \frac{y_1 - y_2}{2+r}$$

For $y_1 = y_2$, HH chooses not to save!

Assume $\beta = 1, r = 0$

- c_1 from (12) simplifies to

$$c_1 = \frac{1}{2}\left[y_1 + y_2\right]$$

- a_2 from (13) simplifies to

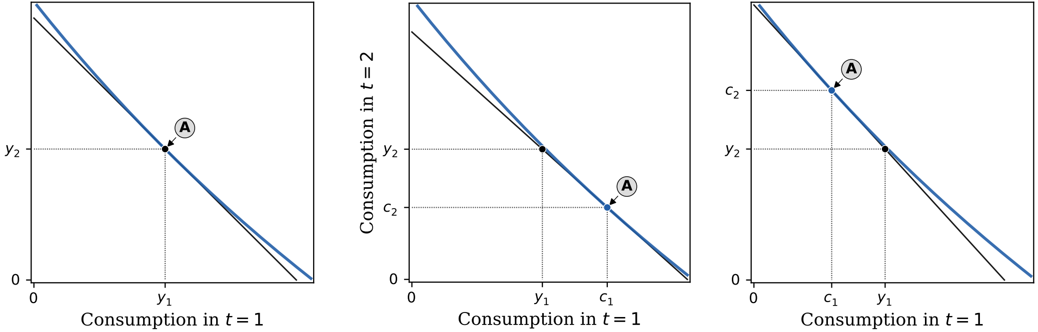
$$a_2 = \frac{1}{2}\left[y_1 - y_2\right]$$

For $y_1 = y_2$, HH chooses not to save!

Optimal intertemporal allocation

Quadratic preferences — Deterministic model

Parameters: $\beta = 1$, $y_1 = y_2 = 1$; utility: $\alpha = 20$, $\delta = 2$



(a) $r = 0\%$ **(b)** $r = -10\%$ **(c)** $r = 10\%$

Figure 3: Intertemporal consumption choice with quadratic preferences and different interest rates. Ⓐ depicts the optimal allocation (c_1, c_2) and the corresponding indifference curve is represented by the blue line.

Quadratic preferences: Stochastic model (certainty equivalence)

Quadratic preferences — Stochastic model

Household problem same as (1), but assume quadratic utility

$$\max_{c_1, c_2, a_2} \left(\alpha c_1 - \frac{\delta}{2} c_1^2\right) + \beta \mathbf{E}\left[\alpha c_2 - \frac{\delta}{2} c_2^2\right]$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = (1+r)a_2 + y_2$$

$$a_2 \geq -b\,, \quad b \equiv \frac{y_{min}}{1+r}$$

where

- a_2 Savings in risk-free bond (not state contingent)
- y_2 Stochastic period-2 income
- y_{min} Lowest possible realisation of y_2, $y_{min} \geq 0$
- b Natural borrowing limit (HH can repay with certainty)

Solving the problem: optimality conditions

Quadratic preferences — Stochastic model

- Euler equation from (5) + (7)

$$\begin{aligned} \alpha - \delta c_1 &= \beta(1+r)\mathbf{E}\left[\,\alpha - \delta c_2\,\right] \\ &= \alpha\beta(1+r) - \delta\beta(1+r)\mathbf{E}c_2 \end{aligned} \tag{14}$$

- Swap expectations and $f(\bullet)$?

$$\mathbf{E}\left[\,f(X)\,\right] \stackrel{?}{=} f(\mathbf{E}X)$$

- Works if f is linear!

$$\mathbf{E}\left[\,f(X)\,\right] = f(\mathbf{E}X)$$

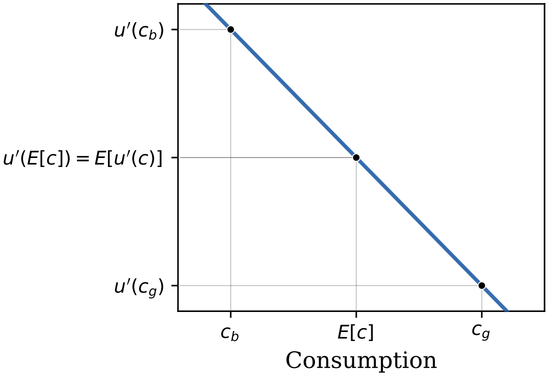
- Apply to quadratic marg. utility (7):

$$\mathbf{E}\left[\,u'(c_2)\,\right] = u'\left(\mathbf{E}c_2\right) = \alpha - \delta\mathbf{E}c_2$$

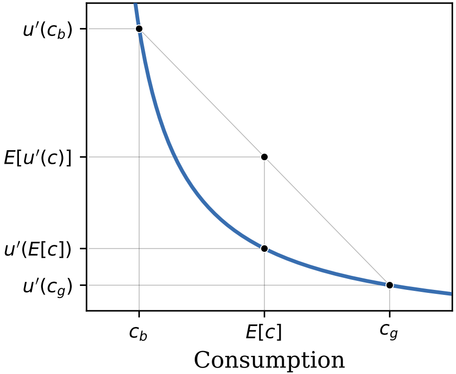
- Does not work with CRRA:

$$\mathbf{E}\left[\,c_2^{-\gamma}\,\right] \neq \left(\mathbf{E}c_2\right)^{-\gamma}$$

Marginal utility: quadratic vs. CRRA preferences



(a) Quadratic utility, $u'(c) = \alpha - \delta c$



(b) Log utility, $u'(c) = 1/c$

Figure 4: Marginal utility for quadratic vs. CRRA preferences. The graph shows a situation in which the consumer faces a gamble with potential outcomes c_b and c_g with equal probability.

Find optimal savings level:

2. Plug budget constraints into EE in (14):

$$\alpha - \delta(y_1 - a_2) = \alpha\beta(1+r) - \delta\beta(1+r)\mathbf{E}\Big[\,(1+r)a_2 + y_2\,\Big]$$

3. Pull a_2 out of expectations:

$$\alpha - \delta y_1 + \delta a_2 = \alpha\beta(1+r) - \delta\beta(1+r)^2 a_2 - \delta\beta(1+r)\mathbf{E}y_2$$

4. Solve for a_2:

$$a_2 = \frac{y_1 - \beta(1+r)\mathbf{E}y_2}{1+\beta(1+r)^2} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \tag{15}$$

5. Use budget constraint, solve for c_1:

$$c_1 = y_1 - a_2 = \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\left[y_1 + \frac{\mathbf{E}y_2}{1+r}\right] + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \tag{16}$$

Solution: deterministic vs. stochastic model

Quadratic preferences

Certainty equivalence: Solutions are identical except for expectations!

Deterministic

Given by (12) and (13):

$$c_1 = \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\left[y_1 + \frac{y_2}{1+r}\right] + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2}$$

$$a_2 = \frac{y_1 - \beta(1+r)y_2}{1+\beta(1+r)^2} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2}$$

Stochastic

Given by (16) and (15)

$$c_1 = \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\left[y_1 + \frac{\mathbf{E}y_2}{1+r}\right] + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2}$$

$$a_2 = \frac{y_1 - \beta(1+r)\mathbf{E}y_2}{1+\beta(1+r)^2} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2}$$

What about c_2? Not the same unless realised $y_2 = \mathrm{E}y_2$

Which economy would the HH prefer? Deterministic economy (due to risk aversion)

Optimal choices vs. interest rate: consumption (left) and assets (right)

Parameters: $\beta = 1$, $y_1 = \mathbf{E}y_2 = 1$; utility: $\alpha = 20$, $\delta = 2$

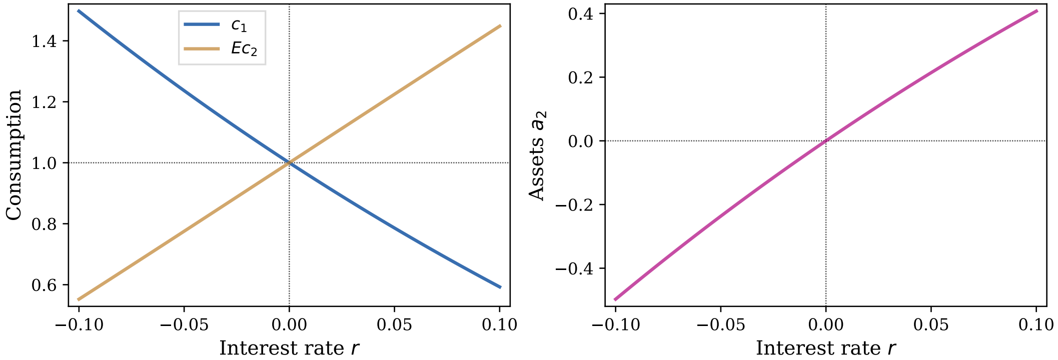


Figure 5: Optional consumption and savings with quadratic utility under uncertainty

Precautionary savings model

Motivation

- Risk has no effect in certainty equivalence model as long mean is the same
- Intuitively, higher risk should trigger precautionary savings response
- Empirical evidence for precautionary savings: HH with more volatile income have higher savings rate
- Quadratic utility unappealing for other reasons (mentioned earlier), rarely used in modern macroeconomics or HH finance
  - Except for some niche applications which we ignore

Need to go back to CRRA preferences to get precautionary savings!

- Problem: hard to solve analytically

Household problem with CRRA preferences

Household problem same as (1), but assume CRRA preferences

$$\max_{c_1, c_2, a_2} \quad u(c_1) + \beta \mathbf{E} u(c_2) \tag{17}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1 \tag{18}$$

$$c_2 = (1+r)a_2 + y_2 \tag{19}$$

$$a_2 \geq -b\,, \quad b \equiv \frac{y_{min}}{1+r} \tag{20}$$

$$y_2 \text{ stochastic with } y_2 \geq y_{min}$$

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases}$$

where

- a_2 Savings in risk-free bond (not state contingent)
- y_2 Stochastic period-2 income
- y_{min} Lowest possible realisation of y_2, $y_{min} \geq 0$
- b Natural borrowing limit (HH can repay with certainty)

Solving the household problem

Precautionary savings model

1. Euler equation: (5) with CRRA marginal utility

$$c_1^{-\gamma} = \beta(1+r)\mathbf{E}\left[c_2^{-\gamma}\right] \qquad (21)$$

- With CRRA we have:

$$\mathbf{E}\left[c_2^{-\gamma}\right] \neq (\mathbf{E}c_2)^{-\gamma}$$

- Strictly convex marginal utility:

$$\mathbf{E}\left[c_2^{-\gamma}\right] > (\mathbf{E}c_2)^{-\gamma}$$

  - Follows from Jensen's inequality
  - Illustrated in Figure 4b
- Compared to certainty equivalence, r.h.s. of EE is larger
  Implication for c_1? $c_1 \downarrow$

Can we solve the household problem with CRRA preferences and uncertainty?

- Express Euler equation in terms of savings a_2:

$$(y_1 - a_2)^{-\gamma} = \beta(1+r)\mathbf{E}\left[\left((1+r)a_2 + y_2\right)^{-\gamma}\right]$$

  Non-linear equation in a_2, no analytical solution!

- Try the usual remedy: log preferences

$$\frac{1}{y_1 - a_2} = \beta(1+r)\mathbf{E}\left[\frac{1}{(1+r)a_2 + y_2}\right] \tag{22}$$

  Still non-linear in a_2, no analytical solution in general!

Solution methods used in the literature

1. Replace terms inside expectations with higher-order Taylor approximation:

   Converts non-linear expression to polynomials in random variables.
2. Make assumptions on joint distribution of consumption, asset returns, etc. to get closed-form solution.

   Consumption taken as exogenous — acceptable in finance but not in macroeconomics!
3. Numerical solution methods

Approach in this unit

Impose sufficiently many simplifying assumptions

Precautionary savings:
Simple model with analytical solution

Simplifying assumptions

- Log preferences ($\gamma = 1$), $\beta = 1$
- Income: $y_1 = \mathbf{E}y_2 = y, \;\; y_2$ with symmetric risk:

$$y_2 = \begin{cases} y - \epsilon & \text{with prob. } \frac{1}{2} \\ y + \epsilon & \text{with prob. } \frac{1}{2} \end{cases} \tag{23}$$

  where $0 < \epsilon < y$
- Borrowing limit: $y_{min} = y - \epsilon$, so

$$a_2 \geq -b\,, \qquad b \equiv \frac{y - \epsilon}{1 + r} > 0$$

  Assume that borrowing limit is not binding

Solving the household problem: optimality conditions

Precautionary savings model

- Euler equation (22) now given by

$$\frac{1}{y-a_2} = (1+r)\underbrace{\left[\frac{1}{2}\frac{1}{(1+r)a_2+y-\epsilon}+\frac{1}{2}\frac{1}{(1+r)a_2+y+\epsilon}\right]}_{\mathbf{E}\left[u'((1+r)a_2+y_2)\right]} \tag{24}$$

- Need to extract a_2 out of expectation
  1. Common denominator:

$$\left[(1+r)a_2+y-\epsilon\right]\left[(1+r)a_2+y+\epsilon\right] = \left[(1+r)a_2+y\right]^2-\epsilon^2$$

  2. Rearrange terms inside bracket of (24)

$$\begin{aligned}\mathbf{E}\left[u'((1+r)a_2+y_2)\right] &= \frac{1}{2}\frac{1}{(1+r)a_2+y-\epsilon}+\frac{1}{2}\frac{1}{(1+r)a_2+y+\epsilon}\\ &= \frac{1}{2}\frac{(1+r)a_2+y+\epsilon}{\left[(1+r)a_2+y\right]^2-\epsilon^2}+\frac{1}{2}\frac{(1+r)a_2+y-\epsilon}{\left[(1+r)a_2+y\right]^2-\epsilon^2}\\ &= \frac{(1+r)a_2+y}{\left[(1+r)a_2+y\right]^2-\epsilon^2}\end{aligned}$$

Solving the household problem: optimality conditions

Precautionary savings model

- Euler equation now reads

$$\frac{1}{y - a_2} = (1+r)\frac{(1+r)a_2 + y}{\left[(1+r)a_2 + y\right]^2 - \epsilon^2} \tag{25}$$

- Expand and collect terms:

$$\underbrace{\left[2(1+r)^2\right]}_{A} a_2^2 + \underbrace{\left[(1+r)(2-r)y\right]}_{B} a_2 + \underbrace{\left[-ry^2 - \epsilon^2\right]}_{C} = 0$$

- Solve using quadratic formula:

$$a_2 = -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

- a_2 as function of parameters:

$$a_2 = -\frac{(2-r)y}{4(1+r)} + \frac{\sqrt{(2+r)^2 y^2 + 8\epsilon^2}}{4(1+r)} \tag{26}$$

Does the solution make sense?

Precautionary savings model

Examine under simplifying assumptions!

Assume $r = 0$

- Solution simplifies to

$$\begin{aligned} a_2 &= -\frac{2y}{4} + \frac{\sqrt{2^2y^2 + 8\epsilon^2}}{4} \\ &> -\frac{2y}{4} + \frac{\sqrt{2^2y^2}}{4} = -\frac{2y}{4} + \frac{2y}{4} \\ &= 0 \end{aligned}$$

- Without uncertainty we know $a_2 = 0$
- With uncertainty, HH saves strictly positive amount
  $\implies$ precautionary savings

Assume $\epsilon = 0$

- a_2 simplifies to

$$\begin{aligned} a_2 &= -\frac{(2-r)y}{4(1+r)} + \frac{\sqrt{(2+r)^2y^2}}{4(1+r)} \\ &= -\frac{(2-r)y}{4(1+r)} + \frac{(2+r)y}{4(1+r)} \\ &= \frac{-2y + ry + 2y + ry}{4(1+r)} \\ &= \frac{1}{2}\frac{r}{1+r}y \qquad (27) \end{aligned}$$

$$\implies \quad c_1 = y - a_2 = \frac{1}{2}\frac{2+r}{1+r}y$$

- Identical to what we found for deterministic model in earlier units

Precautionary savings:
Results from numerical solution

- Relax assumption of log preferences
- Examine increase in ϵ: mean-preserving spread (recall last unit)

$$\mathbf{E}y_2 = \frac{1}{2}(y-\epsilon) + \frac{1}{2}(y+\epsilon) = y$$

$$\text{Var}\,(\,y_2\,) = \frac{1}{2}\left[y-\epsilon-y\right]^2 + \frac{1}{2}\left[y+\epsilon-y\right]^2 = \epsilon^2$$

Effect on precautionary savings? — Can be seen from (26) for log preferences

How does response depend on RRA? — Increasing in RRA

Optimal savings for different RRA and income risk levels

Parameters: $\beta = 1$, $r = 0$. For $\gamma = 1$, this plots optimal a_2 from (26) against ϵ.

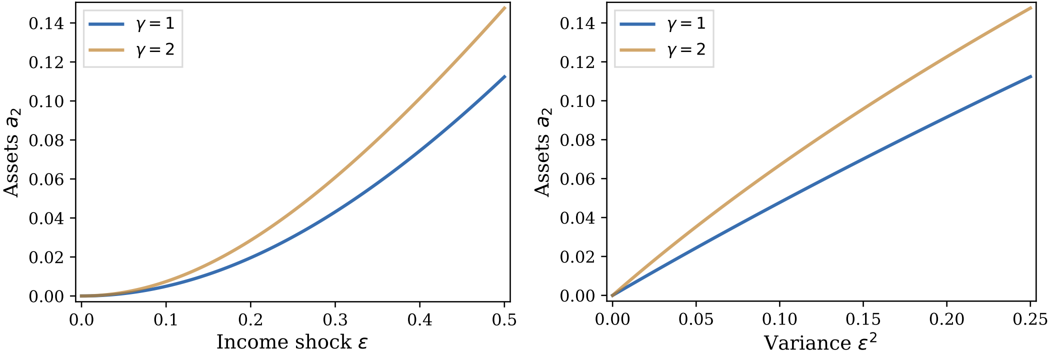


Figure 6: Precautionary savings as a function of the RRA coefficient γ and income risk

Precautionary savings: General equilibrium

- Household problem as before, with CRRA preferences
- $y_1 = \mathbf{E}y_2 = y$, with y_2 given by

$$y_2 = \begin{cases} y - \epsilon & \text{with prob. } \pi \\ y + \epsilon & \text{with prob. } 1 - \pi \end{cases} \tag{28}$$

- Economy populated by arbitrary number of ex ante identical households

How can we solve for equilibrium r?

- HH are ex ante identical $\implies$ all make identical choices c_1, a_2
- Not possible that some HH are savers, others borrowers!
- All HH must consume their endowment each period

Solving for equilibrium

Precautionary savings model

- Euler equation given by

$$c_1^{-\gamma} = \beta(1+r)\mathbf{E}\left[c_2^{-\gamma}\right] \implies y^{-\gamma} = \beta(1+r)\mathbf{E}\left[y_2^{-\gamma}\right]$$

since $c_1 = y$, $c_2 = y_2$

- Expand expectations:

$$y^{-\gamma} = \beta(1+r)\Big[\pi(y-\epsilon)^{-\gamma} + (1-\pi)(y+\epsilon)^{-\gamma}\Big]$$

- Solve for equilibrium r:

$$1 + r = \beta^{-1}\frac{y^{-\gamma}}{\mathbf{E}\left[y_2^{-\gamma}\right]} = \beta^{-1}\frac{y^{-\gamma}}{\pi(y-\epsilon)^{-\gamma} + (1-\pi)(y+\epsilon)^{-\gamma}} \tag{29}$$

- Mean-preserving spread: from Figure 4b we know

$$\epsilon \uparrow \implies \mathbf{E}\left[y_2^{-\gamma}\right] \uparrow \implies r \downarrow$$

Intuition? Riskier income $\implies$ HH wants to increase precautionary savings

Parameters: $y = 1$, $\beta = 1$, $\pi = \frac{1}{2}$. Plots equilibrium r from (29).

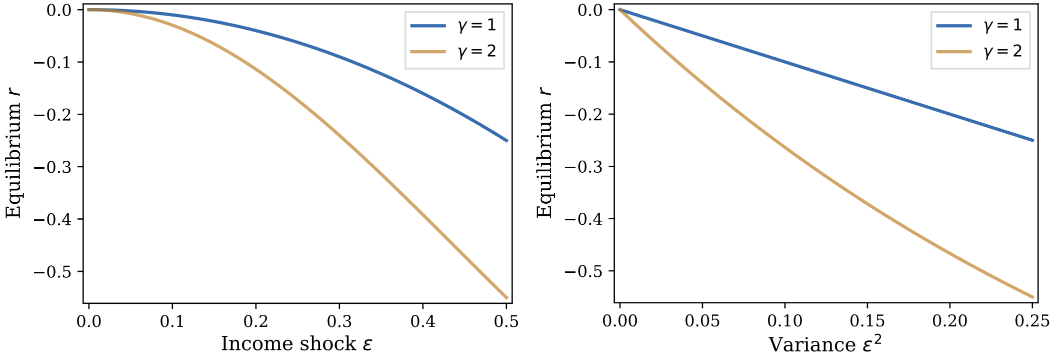


Figure 7: Equilibrium interest rate as a function of income risk and the RRA coefficient γ

Effect of RRA on equilibrium r?

More risk-averse HH wants to increase savings more $\implies r \downarrow$

Main takeaways from this unit

Main takeaways

Certainty equivalence model

1. Optimal choices identical to deterministic case (after replacing certain quantities with expectations)
2. Households do not respond to risk that leaves mean unchanged
3. Allows for analytical solutions, but has many flaws. Rarely used in heterogeneous-agent macroeconomics.

Precautionary savings model

1. Households respond to risk by increasing precautionary savings
   - Savings increasing in shock variance
   - Savings increasing in risk aversion
2. Optimal solutions differ from deterministic counterparts
3. Backbone of modern macroeconomics, but hard to solve analytically

Unit 5: Overlapping generations models

Advanced Macroeconomics (ECON4040) – Part 2

Richard Foltyn

March 17, 2023

Outline for today

In-course exam: March 23, 6:30–9pm

Overlapping generations models (OLG)

Unit 2: lifecycle models

- Analyse choices of single cohort over its lifetime
- Partial equilibrium

Today: Overlapping generations models (OLG)

- Multiple cohorts alive at the same time
- General equilibrium
- Simplest example: two cohorts, each lives for two periods
  - "young" — assumed to work, want to save for retirement
  - "old" — consume savings and die

  Representative cohort: each cohort consists of exactly one household

- Stationary economy exists indefinitely
  - All aggregate quantities are time invariant

Overlapping generations models (OLG)

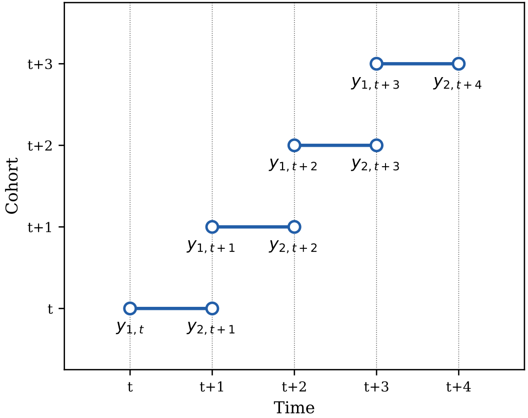


Figure 1: Cohort structure in OLG model with agents who live for two periods. (y_1, y_2) denotes endowments agents receive when young and old, respectively.

Pure endowment economy
with
two cohorts

Pure endowment economy

- Incomplete markets
- Household receives endowment $y_1 > 0$ when young, $y_2 = 0$ when old
- **Maximisation problem:**

$$\max_{c_1, c_2, a_2} \; u(c_1) + \beta u(c_2)$$
$$\text{s.t.} \quad c_1 + a_2 = y_1$$
$$c_2 = (1+r)a_2 \tag{1}$$

- Well-defined problem in partial equilibrium

But does this make sense in general equilibrium?

- Old household:
  - Cannot borrow (not alive to repay)
  - Does not want to save (not alive to consume savings)
- Young household: would like to save
- In aggregate, assets are in zero net supply: sum of saving/borrowing has to be zero

More sensible assumptions for OLG

Need richer environment — Examples?

1. HH receive positive endowments each period
2. Each household lives many periods, many cohorts alive at the same time
   - With many cohorts, young borrow, middle-aged HH save
3. Assets in positive net supply
   1. Government bonds
   2. Production economy with physical capital (not covered in this unit)
4. Government facilitates inter-generational transfers via pension system

Endowments in both periods

- Household receives endowment $y_1 > 0$ when young, $y_2 > 0$ when old
- Maximisation problem:

$$\max_{c_1, c_2, a_2} \quad u(c_1) + \beta u(c_2)$$
$$\text{s.t.} \quad c_1 + a_2 = y_1$$
$$c_2 = (1+r)a_2 + y_2$$

Is there anything new here? — No!

1. Lifetime budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

2. Lagrangian:

$$\mathcal{L} = u(c_1) + \beta u(c_2) + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]$$

3. Euler equation:

$$u'(c_1) = \beta(1+r)u'(c_2)$$

4. Impose autarky: $c_1 = y_1$, $c_2 = y_2$
5. Equilibrium interest rate:

$$r = \frac{u'(y_1)}{\beta u'(y_2)} - 1$$

Pure endowment economy
with
three cohorts

OLG with **three** overlapping generations

- Introduce additional working-age cohort to get around problem of old generation not saving/borrowing

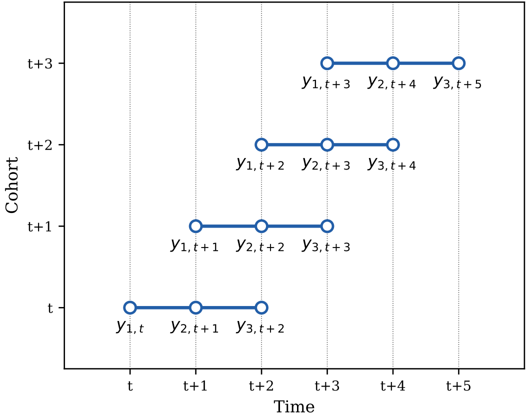


Figure 2: Cohort structure in OLG model with agents who live for three periods..

Household problem

OLG with three cohorts

- **Maximisation problem:**

$$\max_{c_1, c_2, c_3, a_2, a_3} \quad u(c_1) + \beta u(c_2) + \beta^2 u(c_3)$$
$$\text{s.t.} \quad c_1 + a_2 = y_1$$
$$c_2 + a_3 = (1+r)a_2 + y_2$$
$$c_3 = (1+r)a_3 + y_3$$

- Household receives endowments (y_1, y_2, y_3) where y_3 could be zero
- As before: no possibility/incentive to borrow/save in terminal period 3
- Goal: find equilibrium where HH wants to borrow at age 1 and save at age 2.

When will there be such an equilibrium?

- Upward-sloping income trajectory $\Rightarrow$ want to borrow at age 1
- $y_3 \ll y_2$ (low replacement rate) $\Rightarrow$ want to save at age 2

1. Lifetime budget constraint:

$$c_1 + \frac{c_2}{(1+r)} + \frac{c_3}{(1+r)^2} = y_1 + \frac{y_1}{(1+r)} + \frac{y_3}{(1+r)^2}$$

2. Lagrangian:

$$\mathcal{L} = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \lambda\left[y_1 + \frac{y_1}{(1+r)} + \frac{y_3}{(1+r)^2} - c_1 - \frac{c_2}{(1+r)} - \frac{c_3}{(1+r)^2}\right]$$

3. First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta u'(c_2) - \frac{\lambda}{1+r} = 0 \quad (3)$$

$$\frac{\partial \mathcal{L}}{\partial c_3} = \beta^2 u'(c_2) - \frac{\lambda}{(1+r)^2} = 0 \quad (4)$$

4. Euler equations from (2) + (3) and (3) + (4):

$$u'(c_1) = \beta(1+r)u'(c_2)$$
$$u'(c_2) = \beta(1+r)u'(c_3)$$

Solution to the HH problem (partial equilibrium)

OLG with three cohorts

1. Euler equations with CRRA:

$$c_1^{-\gamma} = \beta(1+r)c_2^{-\gamma}$$
$$c_2^{-\gamma} = \beta(1+r)c_3^{-\gamma}$$

2. Express c_2, c_3 in terms of c_1:

$$c_2 = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} c_1$$
$$c_3 = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} c_2 = \left[\beta(1+r)\right]^{\frac{2}{\gamma}} c_1$$

3. Solve for c_1 using LTBC
4. For log preferences, optimal consumption is

$$c_1 = \frac{1}{1+\beta+\beta^2}\left[y_1 + \frac{y_2}{(1+r)} + \frac{y_3}{(1+r)^2}\right] \quad (5)$$
$$c_2 = \frac{\beta(1+r)}{1+\beta+\beta^2}\left[y_1 + \frac{y_2}{(1+r)} + \frac{y_3}{(1+r)^2}\right] \quad (6)$$
$$c_3 = \frac{\beta^2(1+r)^2}{1+\beta+\beta^2}\left[y_1 + \frac{y_2}{(1+r)} + \frac{y_3}{(1+r)^2}\right] \quad (7)$$

- Need to impose market clearing to find r

  Which markets are operational in this economy?
  - Asset market for saving/borrowing

  At which age do HH trade in assets?
  - Borrowing/saving possible at ages 1 + 2
  - No borrowing/saving at age 3

- Market clearing: with representative cohorts, borrowing (savings) at age 1 has to equal savings (borrowing) at age 2:

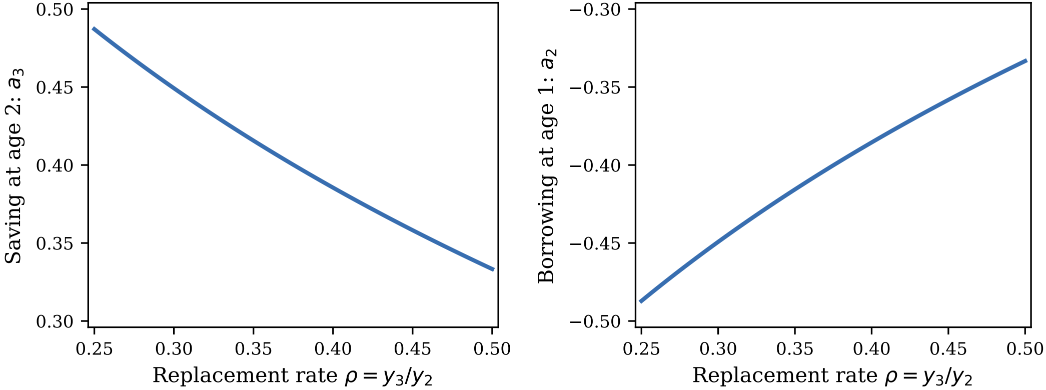
$$-a_2 = a_3$$

- Substitute optimal consumption from (5) and (7) into market clearing condition

$$-\underbrace{(y_1 - c_1)}_{=a_2} = \underbrace{\frac{1}{1+r}(y_3 - c_3)}_{=a_3}$$

  Results in nonlinear equation in r, needs to be solved numerically.

- Numerical solution for $\beta = 1$, $\gamma = 1$, $y_1 = 1$, $y_2 = 2$
- Define replacement rate $\rho = y_3/y_2$



(a) Saving at age 2: a_3

(b) Borrowing at age 1: a_2

Figure 3: Borrowing/saving plotted against the replacement rate $\rho = y_3/y_2$ of retirement income.

Equilibrium interest rate

OLG with three cohorts

- Lower replacement rate ρ increases incentive to save at age 2 (consumption smoothing)
- Lower equilibrium r required so that HH at age 1 is willing to borrow more

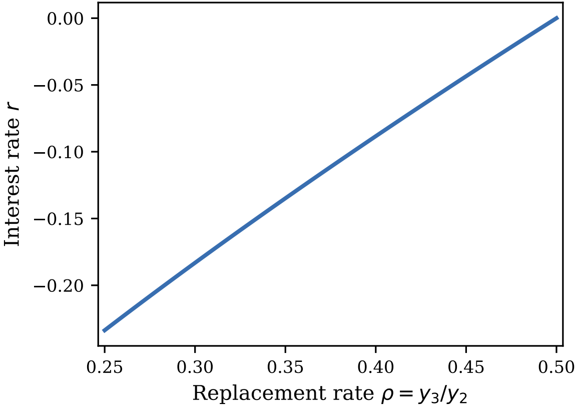


Figure 4: Equilibrium interest rate plotted against the replacement rate $\rho = y_3/y_2$ of retirement income.

Government debt
with
two cohorts

Government debt

- Introduce another agent into economy which supplies savings opportunities
- Infinitely lived government issues debt b_t, pays interest r_t, raises taxes τ_t
- Dynamic government budget constraint:

$$\underbrace{b_{t+1} + \tau_t}_{\text{Revenues}} = \underbrace{(1 + r_t) b_t}_{\text{Debt repayment}}$$

- Stationary economy: b_t, τ_t and r_t constant
  - Government rolls over stock of debt b indefinitely
  - Government budget:

$$b + \tau = (1 + r) b \implies \tau = rb \tag{8}$$

- General equilibrium:
  - Government decides on policy variable b
  - r and τ determined endogenously from (8) and bond market clearing

Household problem (partial equilibrium)

OLG with government debt

- Household problem:

$$\max_{c_1, c_2, a_2} \log(c_1) + \beta \log(c_2)$$
$$\text{s.t.} \quad c_1 + a_2 = y_1 - \tau$$
$$c_2 = (1+r)a_2 \qquad (9)$$

Pays lump sum income tax τ when young

1. Lifetime budget constraint:

$$c_1 + \frac{c_2}{1+r} = y_1 - \tau \qquad (10)$$

2. Euler equation:

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2} \qquad (11)$$

3. Solve (11) for c_1, plug into (10):

$$c_1 + \frac{\beta(1+r)c_1}{1+r} = y_1 - \tau$$
$$\implies c_1 = \frac{1}{1+\beta}\left[y_1 - \tau\right] \qquad (12)$$

4. Optimal savings: (9) + (12)

$$a_2 = y_1 - \tau - c_1 = \frac{\beta}{1+\beta}\left[y_1 - \tau\right] \qquad (13)$$

Which equilibrium conditions need to be satisfied?

1. Bond market clearing: $a_2 = b$
2. Equilibrium r must satisfy HH optimality conditions given disposable income $y_1 - \tau$
3. τ must satisfy government budget constraint (8)

1. Savings: impose $a_2 = b$ in (13):

$$b = \frac{\beta}{1+\beta}\left[y_1 - \tau\right]$$

2. Plug in gov't BC (8):

$$b = \frac{\beta}{1+\beta}\left[y_1 - rb\right]$$

3. Solve for equilibrium r:

$$r = \frac{y_1}{b} - \frac{1+\beta}{\beta} \tag{14}$$

4. Solve for τ from gov't BC:

$$\tau = y_1 - \frac{1+\beta}{\beta}b \tag{15}$$

Equilibrium consumption

- Consumption when young: (12) + (15)

$$c_1 = \frac{1}{1+\beta}\left[y_1 - \tau\right] = \frac{1}{\beta}b \tag{16}$$

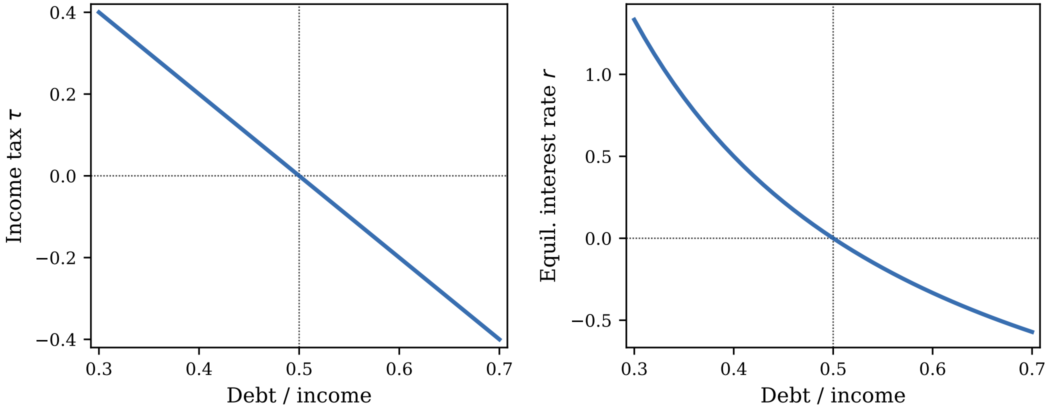
- Consumption when old: (9) + (14)

$$c_2 = (1+r)b = y_1 - \frac{1}{\beta}b \tag{17}$$

Government can set consumption (c_1, c_2) via policy b!

Plot against debt-to-income ratio b/y_1

Each point represents an equilibrium for a given debt level b.



(a) Income tax τ

(b) Interest rate r

Figure 5: Income tax and equilibrium interest rate plotted against the debt-to-income ratio b/y_1 for $\beta = 1$ and $y_1 = 1$.

Plot against debt-to-income ratio b/y_1

Each point represents an equilibrium for a given debt level b.

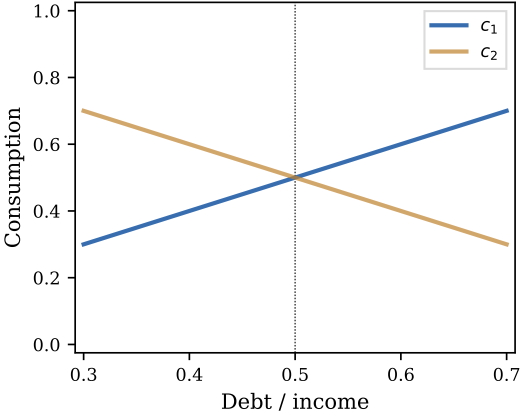


Figure 6: Optimal consumption plotted against the debt-to-income ratio b/y_1 for $\beta = 1$ and $y_1 = 1$.

Optimal level of government debt

Optimal level of government debt

- Which b should the government choose?
  - Takes into account optimal HH response
- Assumption: government values welfare of all cohorts equally
  - Sufficient to maximise utility of one cohort

- **Government problem:**

$$\max_{b \in [0, \beta y_1]} \quad \log(c_1^*) + \beta \log(c_2^*)$$

- c_1^* and c_2^* are optimal HH choices (16) and (17):

$$c_1^* = \frac{1}{\beta} b$$

$$c_2^* = y_1 - \frac{1}{\beta} b$$

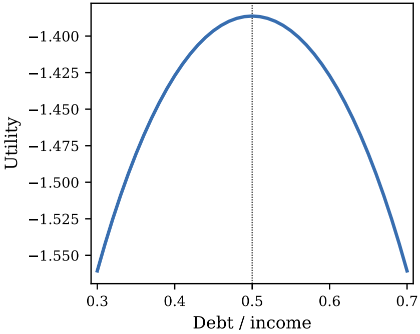


Figure 7: Household utility as a function of debt-to-income ratio for $\beta = 1$ and $y_1 = 1$.

Government problem

OLG with optimal government debt

1. Government objective:

$$\max_{b \in [0, \beta y_1]} \quad \log\left(\beta^{-1} b\right) + \beta \log\left(y_1 - \beta^{-1} b\right)$$

2. First-order condition:

$$\frac{1}{b} - \beta \frac{\beta^{-1}}{y_1 - \beta^{-1} b} = 0 \qquad (18)$$

3. Solve (18) for b:

$$b^* = \frac{\beta}{1+\beta} y_1 \qquad (19)$$

4. Welfare-maximising c_1:

$$c_1 = \frac{1}{\beta} b^* = \frac{1}{1+\beta} y_1$$

5. Welfare-maximising c_2:

$$c_2 = y_1 - \frac{1}{\beta} b^* = \frac{\beta}{1+\beta} y_1$$

6. Equilibrium interest rate: (14) + (19)

$$r = \frac{y_1}{\frac{\beta}{1+\beta} y_1} - \frac{1+\beta}{\beta} = 0 \qquad (20)$$

Equilibrium illustrated by dotted lines in Figure 5, Figure 6 and Figure 7

Pension system
with exogenous labour supply

Pension system with exogenous labour supply

- Alternative way to transfer resources between cohorts: pension system
- PAYGO: pay-as-you-go pension system
  - Government imposes payroll tax τ on working (young) households
  - Distributes pension payments T to old

Budget balance (assuming one HH per cohort):

$$\underbrace{\tau}_{\text{Payroll tax revenues}} = \underbrace{T}_{\text{Pensions}}$$

1. Household problem

$$\max_{c_1, c_2, a_2} \quad u(c_1) + \beta u(c_2)$$
$$\text{s.t.} \quad c_1 + a_2 = y_1 - \tau$$
$$c_2 = (1+r)a_2 + \tau$$

2. Euler equation is standard:

$$u'(c_1) = \beta(1+r)u'(c_2) \qquad (21)$$

3. No saving in equilibrium:

$$c_1 = y_1 - \tau$$
$$c_2 = \tau$$

4. Equilibrium r from (21):

$$r = \frac{u'(y_1 - \tau)}{\beta u'(\tau)} - 1$$

Again, HH consumption is fully determined by government policy τ!

Which τ should government implement?

1. Government objective (CRRA):

$$\max_{\tau \in [0, y_1]} \frac{(y_1 - \tau)^{1-\gamma}}{1-\gamma} + \beta \frac{\tau^{1-\gamma}}{1-\gamma}$$

2. First-order condition:

$$-(y_1 - \tau)^{-\gamma} + \beta \tau^{-\gamma} = 0$$

3. Optimal τ:

$$\tau = \frac{y_1}{1 + \beta^{-\frac{1}{\gamma}}} \qquad (22)$$

4. Welfare-maximising consumption:

$$c_1 = y_1 - \tau = \frac{\beta^{-\frac{1}{\gamma}}}{1 + \beta^{-\frac{1}{\gamma}}} y_1$$

$$c_2 = \tau = \frac{1}{1 + \beta^{-\frac{1}{\gamma}}} y_1$$

5. Equilibrium r from EE (21):

$$\left(\frac{\beta^{-\frac{1}{\gamma}}}{1 + \beta^{-\frac{1}{\gamma}}} y_1 \right)^{-\gamma} = \beta(1+r) \left(\frac{1}{1 + \beta^{-\frac{1}{\gamma}}} y_1 \right)^{-\gamma}$$

$$\beta = \beta(1+r)$$

$$\implies \quad r = 0$$

Optimal payroll tax: Intuition

Pension system with exogenous labour supply

Simplifying assumptions to get some intuition

Assume $\beta = 1$

- Optimal payroll tax:

$$\tau = \frac{1}{2} y_1$$

  Half of endowment consumed in each period

Assume $\gamma = 1$

- Optimal payroll tax:

$$\tau = \frac{1}{1 + \beta^{-1}} y_1 = \frac{\beta}{1 + \beta} y_1$$

  Identical to optimal savings if savings was possible

- Optimal consumption c_1:

$$c_1 = y_1 - \tau = \frac{1}{1 + \beta} y_1$$

Pension system
with endogenous labour supply

Pension system with endogenous labour supply

- So far, labour supply was exogenous (= endowment)
- Payroll taxes could potentially affect willingness to work
  - Effect on aggregate output in production economy?

Economic environment

- Endogenous leisure choice ℓ, labour supply $1 - \ell$
- Production function $f(L) = A \cdot L$
  - Implies equilibrium wage $w = A$
- Proportional payroll tax τ
- Lump-sum pension transfer T
- Government budget balance (PAYGO):

$$\underbrace{T}_{\text{Pensions}} = \underbrace{\tau w(1 - \ell)}_{\text{Payroll taxes}} \tag{23}$$

Household problem

Pension system with endogenous labour supply

- Household maximises:

$$\max_{c_1, c_2, a_2} \quad \log(c_1) + \log(\ell) + \beta \log(c_2)$$
$$\text{s.t.} \quad c_1 + a_2 = (1-\tau)w(1-\ell) \tag{24}$$
$$c_2 = (1+r)a_2 + T \tag{25}$$

- Supplies labour $0 \leq 1-\ell \leq 1$ while young
- Receives pension T when old
- log-log preferences like in part 1 of the course
- Lifetime budget constraint:

$$c_1 + \frac{c_2}{1+r} = (1-\tau)w(1-\ell) + \frac{T}{1+r} \tag{26}$$

Household optimality conditions

Pension system with endogenous labour supply

1. Lagrangian:

$$\mathcal{L} = \log(c_1) + \log(\ell) + \beta \log(c_2) + \lambda \left[(1-\tau)w(1-\ell) + \frac{T}{1+r} - c_1 + \frac{c_2}{1+r} \right]$$

2. First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \tag{27}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta \frac{1}{c_2} - \frac{\lambda}{1+r} = 0 \tag{28}$$

$$\frac{\partial \mathcal{L}}{\partial \ell} = \frac{1}{\ell} - \lambda(1-\tau)w = 0 \tag{29}$$

3. Euler equation: (27) + (28)

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2}$$

4. Intra-temporal optimality: (27) + (29)

$$\underbrace{\frac{1/\ell}{1/c_1}}_{MRS_{c_1,\ell}} = \underbrace{\frac{(1-\tau)w}{1}}_{\text{Relative price}} \tag{30}$$

Solve for c_1:

$$c_1 = \ell(1-\tau)w \tag{31}$$

Solving for equilibrium

Pension system with endogenous labour supply

1. No savings in equilibrium: Set $a_2 = 0$ in (24) + (25) + (23)

$$c_1 = (1-\tau)w(1-\ell) \tag{32}$$

$$c_2 = T = \tau(1-\ell)w \tag{33}$$

2. Combine (31) + (32):

$$\begin{aligned} c_1 &= (1-\tau)w(1-\ell) \\ \ell(1-\tau)w &= (1-\tau)w(1-\ell) \\ \ell &= (1-\ell) \\ \implies \quad \ell &= \frac{1}{2} \end{aligned}$$

3. Optimal consumption:

$$\begin{aligned} c_1 &= \frac{1}{2}(1-\tau)w \\ c_2 &= \frac{1}{2}\tau w \end{aligned}$$

4. Equilibrium interest rate from EE:

$$\begin{aligned} \frac{1}{\frac{1}{2}(1-\tau)w} &= \beta(1+r)\frac{1}{\frac{1}{2}\tau w} \\ \frac{1}{1-\tau} &= \beta(1+r)\frac{1}{\tau} \\ \implies \quad r &= \frac{1}{\beta}\frac{\tau}{1-\tau} - 1 \end{aligned} \tag{34}$$

1. Government solves:

$$\max_{\tau \in [0,1]} \log(c_1^*) + \log(\ell^*) + \beta \log(c_2^*)$$

2. Plug in HH choices:

$$\max_{\tau \in [0,1]} \log\left(\frac{1}{2}(1-\tau)w\right) + \log\left(\frac{1}{2}\right) + \beta \log\left(\frac{1}{2}\tau w\right)$$

3. Equivalent problem (for fixed $w = A$):

$$\max_{\tau \in [0,1]} \log((1-\tau)) + \beta \log(\tau)$$

4. First-order condition:

$$-\frac{1}{1-\tau} + \beta \frac{1}{\tau} = 0 \tag{35}$$

5. Welfare-maximising τ:

$$\tau = \frac{\beta}{1+\beta}$$

6. Equilibrium interest rate from EE:

$$r^* = \frac{1}{\beta} \frac{\frac{\beta}{1+\beta}}{1 - \frac{\beta}{1+\beta}} - 1$$

$$= \frac{1}{\beta} \frac{\frac{\beta}{1+\beta}}{\frac{1}{1+\beta}} - 1 = \frac{1}{\beta} \frac{\beta}{1} - 1 = 0$$

Comparing models with government

Consumption allocation and r with optimal government policy and log preferences

Government debt

- Interest rate: $r = 0$
- Consumption:

$$c_1 = \frac{1}{1+\beta} y_1$$
$$c_2 = \frac{\beta}{1+\beta} y_1$$

Pensions + exog. labour

- Interest rate: $r = 0$
- Consumption:

$$c_1 = \frac{1}{1+\beta} y_1$$
$$c_2 = \frac{\beta}{1+\beta} y_1$$

Pensions + endog. labour

- Interest rate: $r = 0$
- Consumption:

$$c_1 = \frac{1}{1+\beta} \frac{1}{2} w$$
$$c_2 = \frac{\beta}{1+\beta} \frac{1}{2} w$$

With endog. labour: set productivity $A = 2y_1 = w$ to get identical allocation

Social planner solution

Social planner problem

- Weighted maximisation:

$$\max_{c_1, c_2} \log(c_1) + \beta \log(c_2)$$
$$\text{s.t.} \quad c_1 + c_2 = y_1$$

Attaches weight 1 to young, weight β to old

1. First-order conditions:

$$\frac{1}{c_1} = \lambda$$
$$\beta \frac{1}{c_2} = \lambda$$

1. Cohort-specific consumption linked by

$$\frac{1}{c_1} = \beta \frac{1}{c_2}$$
$$\Longrightarrow \quad c_2 = \beta c_1$$

2. Plug into resource constraint:

$$c_1 + \beta c_1 = y_1$$
$$\Longrightarrow \quad c_1 = \frac{1}{1+\beta} y_1$$
$$\Longrightarrow \quad c_2 = \frac{\beta}{1+\beta} y_1$$

Conclusion: we have solved the same problem three times! — Government can use single policy variable to achieve first best

Main takeaways from this unit

Main takeaways

Endowment economy

1. With only two cohorts and no government, autarky is only achievable equilibrium (in incomplete markets)
2. Government can help transfer resources between cohorts / across time:
   - Government debt: asset in positive net supply
   - Pension system (with exogenous or endogenous labour supply)
3. More than two cohorts: young can borrow from middle-aged HH who save for retirement

More complex OLG models

1. Production economy with capital: savings possible even with two cohorts, no government needed