Advanced Macroeconomics (ECON4040)

Part 2

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March 16, 2023

Contents

1 Heterogeneity in macroeconomics

1.1 Introduction

Representative-agent (RA) models are ubiquitous in macroeconomics. This is in part due to historical reasons as they are easier to solve than heterogeneous-agent models, since the latter usually require numerical solution methods that were not feasible a few decades ago. If one is only interested in aggregate outcomes (such as prices, aggregate consumption, GDP, etc.), in many cases RA models can be used without loss of generality if a model economy *perfectly aggregates*. Loosely speaking, aggregation holds if aggregate outcomes are the same irrespective of whether we model a single, representative agent or a group of agents which are allowed to differ in their wealth, income, labour productivity, age or some other characteristic. Conversely, if an economy with heterogeneous agents behaves very differently in the aggregate, a representative-agent economy may not be adequate to explain an empirical observation or evaluate a government policy. Moreover, RA models by construction have nothing to say about issues of inequality (e.g., in terms of consumption, income or wealth), of which there is plenty as we will see in section 1.5.

We will not be concerned with the formal requirements for aggregation to hold, but will instead illustrate the concept using a few examples. Aggregation usually fails if markets are not *complete* which means that households cannot perfectly insure against their *idiosyncratic* risk, i.e., risk that affects only a single household, as opposed to the entire economy. However, aggregation may fail even in the absence of risk, for example if households face borrowing constraints. We discuss this case below.

First, however, we start by going over the two-period consumption-savings problem in an economy with either one or two households, which will serve as our modelling framework for the rest of this unit (and for most of the course).

1.2 Consumption-savings with heterogeneous households

To fix ideas, recall the two-period household maximisation problem which you already encountered:

$$\max_{c_1,\, c_2,\, a_2} \quad u(c_1) + \beta u(c_2) \tag{1.1}$$

$$\begin{aligned} \text{s.t.} \quad c_1 + a_2 &= a_1 + y_1 \\ c_2 &= (1+r)a_2 + y_2 \\ c_1 \geq 0,&\ c_2 \geq 0 \end{aligned} \tag{1.2}$$

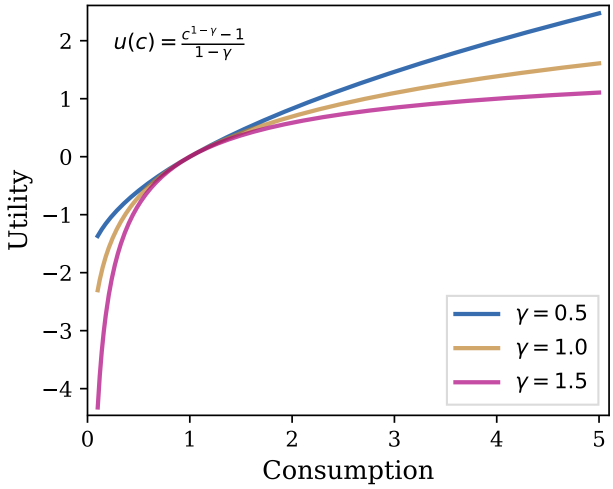


Figure 1.1: CRRA utility for different values of the relative risk aversion parameter γ.

Throughout these notes, we use a utility function that exhibits *constant relative risk aversion* (CRRA), the most common type of household preferences used in macroeconomics:

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases} \tag{1.3}$$

In the previous part, you used log preferences which are a special case if the relative risk aversion parameter γ is set to one.[1] We discuss the exact interpretation of the RRA parameter later, for now it is sufficient to think of it as governing the curvature of the utility function (see Figure 1.1).

The agent enters the first period with assets a_1, receives income (y_1, y_2) in the two periods and can borrow or save at a fixed interest rate r which is taken as given by the household. Consumption has to be non-negative as required by (1.2), but given our usual choice of utility function, the household will always choose positive consumption, so we ignore these constraints from now on.[2] To keep the exposition simple, we for now impose log preferences, $u(c) = \log(c)$, no discounting so that $\beta = 1$, and we assume that initial assets are zero, $a_1 = 0$. We will return to a more general formulation at a later point. With these simplifications, the household problem reads

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \log(c_2) \tag{1.4}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1 \tag{1.5}$$

$$c_2 = (1+r)a_2 + y_2 \tag{1.6}$$

We now proceed to solve for the optimal choices and for general equilibrium under two different assumptions on borrowing.

[1] Using L'Hospitals rule, one can show that $\lim_{\gamma\to 1} \frac{c^{1-\gamma}-1}{1-\gamma} = \log(c)$. The additional constant $-\frac{1}{1-\gamma}$ does not affect the maximisation problem so it is often omitted and utility is written as $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$.

[2] With CRRA preferences, utility $u(c)$ approaches $-\infty$ as c approaches zero, so this can never be optimal.

Example 1.1 (Two-period problem without borrowing constraints). Consider the household problem in (1.4). Without borrowing constraints, the household can choose any a_2 (even negative values!), and we can therefore combine the per-period budget constraints (1.5) and (1.6) into a single *present-value lifetime budget constraint* by solving (1.5) for $a_2 = y_1 - c_1$ and plugging this expression into (1.6):

$$c_2 = (1+r)\big(y_1 - c_1\big) + y_2 \ .$$

Dividing by $(1+r)$ and collecting the consumption terms on one side, we see that

$$\underbrace{c_1 + \frac{c_2}{1+r}}_{\text{PV of cons.}} = \underbrace{y_1 + \frac{y_2}{1+r}}_{\text{PV of income}} \ . \tag{1.7}$$

This lifetime budget constraint has an intuitive interpretation: because households can shift consumption across periods as they see fit (no borrowing constraint), it only requires that the discounted sum of lifetime consumption on the left-hand side equals the discounted sum of lifetime income on the right-hand side.

The Lagrangian for this problem is given by

$$\mathcal{L} = \log(c_1) + \log(c_2) + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r}\right] \tag{1.8}$$

where $\lambda \geq 0$ is the Lagrange multiplier associated with the lifetime budget constraint (1.7). The first-order conditions for c_1 and c_2 are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_1} &= \frac{1}{c_1} - \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial c_2} &= \frac{1}{c_2} - \frac{\lambda}{1+r} = 0
\end{aligned}$$

We combine these to eliminate λ and obtain the *Euler equation*:

$$\frac{1}{c_1} = (1+r)\frac{1}{c_2} \tag{1.9}$$

This intertemporal optimality condition states that at the optimum, the household cannot do better by shifting resources between periods because the marginal gain or loss in period 1 is exactly offset in period 2.

We can solve (1.9) for c_2,

$$c_2 = (1+r)c_1 \ , \tag{1.10}$$

and substitute the expression into the budget constraint (1.7):

$$c_1 + \frac{(1+r)c_1}{1+r} = y_1 + \frac{y_2}{1+r}$$

Solving for c_1, we see that optimal first-period consumption is

$$c_1 = \frac{1}{2}\left[y_1 + \frac{y_2}{1+r}\right] \ . \tag{1.11}$$

The household consumes exactly half of its discounted lifetime income in the first period, irrespective of the interest rate.[3] To find optimal period-2 consumption, we plug (1.11) into (1.10):

$$c_2 = \frac{1}{2}\Big[(1+r)y_1 + y_2\Big] \tag{1.12}$$

Equations (1.11) and (1.12) constitute the solution to the household's optimisation problem. ■

The previous example demonstrated how to solve the household problem in *partial equilibrium* — the interest rate was taken as given and so far our model had nothing to say about its general equilibrium value. To make further progress, we need to make additional assumptions about the structure of the economy. In what follows, we investigate two cases: an equilibrium with two types of households (the simplest form of heterogeneity) where we allow for borrowing, and one where we do not.

Example 1.2 (General equilibrium with borrowing). Continuing with Example 1.1, we now assume that there are two households in this economy, labelled A and B. Both have identical preferences but differ in their income streams:

$$y_1^A = 3,\ y_2^A = 1$$
$$y_1^B = 1,\ y_2^B = 3$$

Intuitively, since utility is concave, these households will want to trade to smooth their consumption across both periods instead of consuming their varying per-period income. All we need to do is find the equilibrium interest rate that clears markets. Recall that a (general) equilibrium is defined as a collection of:

1. Optimal consumption rules for each period which are functions of (y_1, y_2) and the interest rate r. We derived these in (1.11) and (1.12).

2. Equilibrium prices which clear markets. In this example, we have only a single price, r, since we normalised the price of period-1 consumption to 1.

Before finding the market-clearing r, we should remind ourselves which markets need to be cleared. In this endowment economy, there are only two markets:

1. The market for consumption in period 1 (or equivalently, the market for savings in period 1).

2. The market for consumption in period 2.

Because of Walras' law, we need to clear only on of these, and market clearing in the other will obtain automatically. We can thus proceed in two equivalent ways: first, market

[3] Of course, lifetime income itself depends on the interest rate, so the level of c_1 also depends on r.

clearing for savings in the first period requires that savings by A equal borrowing by B, and therefore the following equality has to hold:

$$\underbrace{y_1^A - c_1^A}_{\text{Savings by } A} = \underbrace{c_1^B - y_1^B}_{\text{Borrowing by } B} \tag{1.13}$$

We can substitute for optional consumption using (1.11) to find

$$\begin{aligned} y_1^A - c_1^A &= c_1^B - y_1^B \\ y_1^A - \frac{1}{2}\left[y_1^A + \frac{y_2^A}{1+r}\right] &= \frac{1}{2}\left[y_1^B + \frac{y_2^B}{1+r}\right] - y_1^B \\ y_1^A + y_1^B &= \frac{y_2^A + y_2^B}{1+r} \\ Y_1 &= \frac{Y_2}{1+r} \end{aligned}$$

On the last line, we define *aggregate income* in period t as $Y_t = y_t^A + y_t^B$. The equilibrium interest rate is therefore obtained as

$$r = \frac{Y_2}{Y_1} - 1\,. \tag{1.14}$$

In our example, $Y_1 = Y_2 = 4$, so $r = 0$. The discounted lifetime income for both types $i = A, B$ is therefore

$$y_1^i + \frac{y_2^i}{1+r} = y_1^i + y_2^i = 4\,. \tag{1.15}$$

Finally, (1.11) and (1.12) imply that both households consume two units in each period, so they perfectly smooth consumption:

$$c_1^A = c_2^A = c_1^B = c_2^B = 2\,. \tag{1.16}$$

The other way to compute the equilibrium interest rate is to make sure that the goods market in the second period clears (which is the same as making sure that the aggregate resource constraint holds):

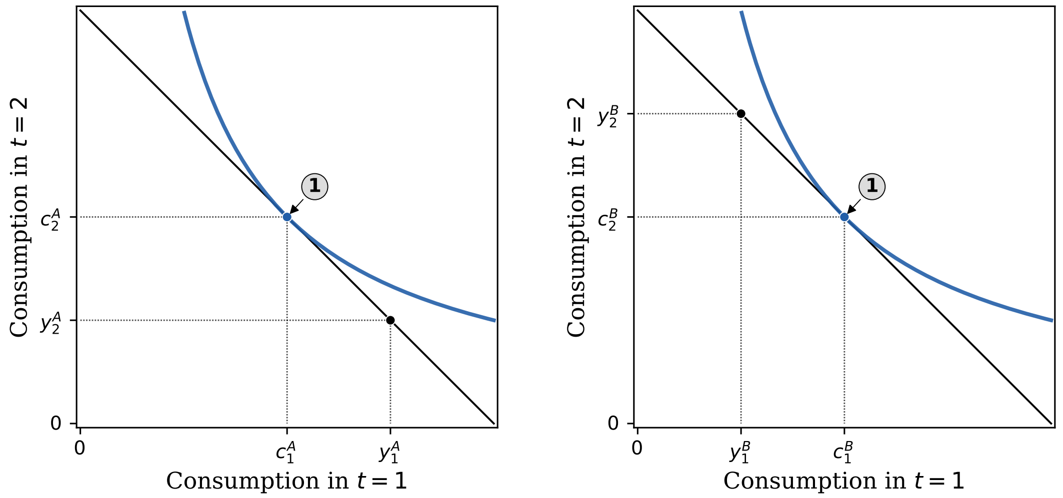
$$\underbrace{c_2^A + c_2^B}_{\text{Aggregate consumption}} = \underbrace{y_2^A + y_2^B}_{\text{Aggregate endowment}}$$

Since there is no savings technology such as physical capital in this economy, the amount consumed in the aggregate must equal the aggregate endowment in each period. If we plug in the optimal consumption rules and do the algebra, we arrive at the same interest rate as in (1.14).

Figure 1.2 illustrates this equilibrium graphically. The budget line has a slope of $-(1+r) = -1$ since the interest rate is zero, and it is a tangent to the indifference curve at the optimal consumption allocation at point 1.[4]

■

[4]Recall that an indifference curve is the collection of all consumption bundles (c_1, c_2) which yield the same utility $\overline{U}$, i.e., it is implicitly defined by $\overline{U} = \log(c_1) + \log(c_2)$.



(a) Type A **(b)** Type B

Figure 1.2: General equilibrium in Example 1.2 *with* borrowing. (1) shows the equilibrium allocation and the blue lines are the corresponding indifference curves.

In the above example, type B has to borrow one unit in the first period to achieve the desired consumption bundle. How does the equilibrium change if we shut down borrowing completely (but the option to save remains intact)? We discuss this case next.

Example 1.3 (General equilibrium without borrowing). We now introduce a slight modification to the household problem in (1.4):

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \log(c_2) \tag{1.17}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$
$$c_2 = (1+r)a_2 + y_2$$
$$a_2 \geq 0 \tag{1.18}$$

Here we added the constraint (1.18) so that the household can no longer borrow. Let's again assume that we have two types of households, A and B, and they have the same income streams as in Example 1.2:

$$y_1^A = 3,\ y_2^A = 1$$
$$y_1^B = 1,\ y_2^B = 3$$

We can solve this problem in two different ways:

1. We use economic intuition and the results we found in Example 1.2 to quickly identify the solution without too much math (the "shortcut").
2. We solve the full constrained optimisation problem in (1.17).

We'll work through both approaches in turn.

Shortcut solution method. Previously, we found that household B was a borrower in equilibrium, which is no longer permitted. While saving is still allowed and household A would want to do so, we no longer have a counterpart in the economy where these savings could be channelled (we say that the savings asset is in *zero net supply*, i.e., all saving and borrowing must sum to zero in the aggregate). We therefore need to find an equilibrium interest rate at which A is content to no longer save but instead just consumes its endowment each period.

Since this interest rate has to be compatible with A's optimality conditions, the natural way to find it is to evaluate A's Euler equation (1.9) at $c_1^A = y_1^A$ and $c_2^A = y_2^A$:

$$\frac{1}{c_1^A} = (1+r)\frac{1}{c_2^A} \implies \frac{1}{y_1^A} = (1+r)\frac{1}{y_2^A} \implies r = \frac{y_2^A}{y_1^A} - 1$$

Plugging in A's income, we find that in equilibrium,

$$r = \frac{y_2^A}{y_1^A} - 1 = \frac{1}{3} - 1 = -\frac{2}{3} \approx -66.7\%$$

As far as interest rates go, this one is quite low. The reason is that household A would really want to save because its income in the first period is so much larger. To discourage A from doing so in equilibrium, the required interest rate has to be sufficiently unattractive!

Constrained optimisation problem. We now turn to the more formal way to derive this result. With the additional constraint, we need to be somewhat more careful how we set up the Lagrangian. Just adding the inequality constraint $a_2 \geq 0$ to (1.8) will not work because we eliminated a_2 as a choice variable by using the consolidated lifetime budget constraint! Consequently, one solution is to add each per-period budget constraint separately and keep a_2 as a choice variable, giving us the following Lagrangian:[5]

$$\begin{aligned}\mathcal{L} = \log(c_1) + \log(c_2) &+ \lambda_1 \underbrace{\Big[y_1 - a_2 - c_1\Big]}_{\text{Budget constr. } t=1} \\ &+ \lambda_2 \underbrace{\Big[(1+r)a_2 + y_2 - c_2\Big]}_{\text{Budget constr. } t=2} \\ &+ \lambda_a \cdot \underbrace{a_2}_{\text{Borrowing constr.}} \end{aligned} \tag{1.19}$$

As there are three different constraints, we have three Lagrange multipliers $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\lambda_a \geq 0$. You may not have encountered inequality constraints like the one

[5] An alternative approach is to eliminate c_1 and c_2, define the Lagrangian as

$$\mathcal{L} = \log(y_1 - a_2) + \log\big((1+r)a_2 + y_2\big) + \lambda_a a_2$$

and take the derivative with respect to a_2. The resulting Euler equation is identical.

in (1.18), so let's briefly discuss how such constraints enter the Lagrangian. If we need to impose an inequality constraint $x \geq y$, we write it as $x - y \geq 0$ and add the term $\lambda(x - y)$ to the Lagrangian where $\lambda \geq 0$ is the multiplier for this constraint.

To find the optimum, we take derivatives as usual, in this case with respect to the choice variables c_1, c_2 and a_2:

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda_1 = 0 \tag{1.20}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \frac{1}{c_2} - \lambda_2 = 0 \tag{1.21}$$

$$\frac{\partial \mathcal{L}}{\partial a_2} = -\lambda_1 + \lambda_2(1 + r) + \lambda_a = 0 \tag{1.22}$$

However, in order to deal with the inequality constraint, there is an additional so-called *complementary slackness condition*, $\lambda_a \cdot a_2 = 0$, which says the following:

1. Either the constraint is binding, i.e., $a_2 = 0$ and $\lambda_a \geq 0$.

2. The constraint is not binding, hence $a_2 > 0$ and complementary slackness requires that $\lambda_a = 0$. This makes sense if you recall the interpretation of the Lagrange multiplier: it tells us how much the objective changes if we relax the constraint by one unit. If the constraint is not binding, however, relaxing it further will not change the optimal choice and will thus not alter the objective, so the value of the multiplier must be zero!

To get the Euler equation for this problem, we substitute for λ_1 and λ_2 in (1.22) using (1.20) and (1.21). This yields

$$\frac{1}{c_1} = (1 + r)\frac{1}{c_2} + \lambda_a \tag{1.23}$$

which is almost identical to the Euler equation in (1.9) when borrowing was allowed. The interpretation is as follows: if the household does not want to borrow at the optimum, then $\lambda_a = 0$ and we are back to the original Euler equation. If, on the other hand, the household would want to borrow but is not allowed to do so, then $\lambda_a > 0$ and the Euler equation is no longer informative (we usually don't know the value of λ_a).

Finding the equilibrium is somewhat more difficult than in Example 1.2 because whether a household wants to borrow or not depends on the interest rate, but the interest rate itself depends on the households' desire to borrow or save. We thus do not know ex ante whether the borrowing constraint is binding or not. We therefore employ an approach called "guess and verify:" we make a conjecture about the equilibrium allocation and try to find an interest rate that supports it. If such an interest rate exists, our conjecture indeed is an equilibrium.

1. Previously, we found that household B would want to borrow, which is no longer possible. We therefore conjecture that in equilibrium, B will be exactly at the borrowing constraint and $\lambda_a^B > 0$.

   Note that B's Euler equation thus contains two unknowns, r and λ_a^B, and therefore will not be helpful to pin down the equilibrium interest rate.

2. It would be reasonable to conjecture that household A still wants to save to smooth consumption, but now there is no counterparty in the economy to absorb these savings. Consequently, A is also forced to consume its endowment so that $c_1^A = y_1^A = 3$ and $c_2^A = y_2^A = 1$. However, unlike B, household A had no intention to borrow in the unconstrained problem, so $\lambda_a^A = 0$.

Under these assumptions, we can use A's Euler equation to find the equilibrium interest rate just like in the "shortcut" approach:

$$\frac{1}{y_1^A} = (1+r)\frac{1}{y_2^A} \implies r = \frac{y_2^A}{y_1^A} - 1$$

To support this equilibrium allocation, we require that

$$r = \frac{1}{3} - 1 = -\frac{2}{3} \approx -66.7\%$$

As a last step, we need to verify whether household B is borrowing constrained given the equilibrium interest rate, as initially conjectured. Plugging into B's Euler equation, we find that

$$\begin{aligned} \frac{1}{c_1^B} = (1+r)\frac{1}{c_2^B} + \lambda_a^B &\implies \frac{1}{1} = \left(1 - \frac{2}{3}\right)\frac{1}{3} + \lambda_a^B \\ &\implies 1 = \frac{1}{9} + \lambda_a^B \\ &\implies \lambda_a^B = \frac{8}{9} > 0 \end{aligned}$$

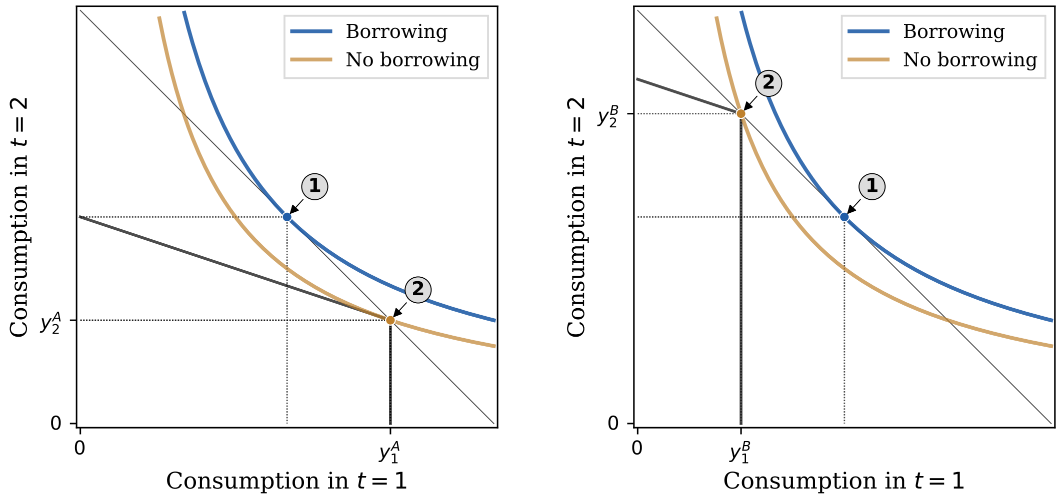
To summarise, we assumed that B would want to borrow if allowed, used this to find the equilibrium interest rate r, and verified that at that interest rate, B would indeed want to borrow because the Lagrange multiplier on the borrowing constraint is positive.

Figure 1.3 illustrates this equilibrium graphically and contrasts it with what we found in Example 1.2. Note that the budget line now has a kink at the period-1 income which arises due to the lack of borrowing opportunities, and the indifference curves indicate that both households are worse off. While we did allow for saving, no household chose to do so at the prevailing interest rate. The resulting equilibrium allocation is called *autarky* because each household just consumes its endowment and does not trade with others.

■

1.3 Aggregation

We now turn to the issue of aggregation: can the equilibria discussed in Example 1.2 and Example 1.3 be modelled using a representative agent? It turns out that this is true in one case, but not in the other.



(a) Type A

(b) Type B

Figure 1.3: Equilibrium for Example 1.3 *without* borrowing. (1) shows the unattainable allocation with borrowing, while (2) is the new autarky allocation. The thick black line depicts the budget line without borrowing, the blue line the indifference curve with borrowing, and the yellow line the indifference curve without borrowing.

Example 1.4 (Aggregation with borrowing). Continuing with Example 1.2, we now ask whether this economy aggregates. If we had a representative agent (RA) endowed with the aggregate income, would we observe the same aggregate consumption and interest rate in equilibrium?

Recall that aggregate income in this economy is given by $Y_1 = Y_2 = 4$. Assuming that the representative agent has the same preferences, the household problem is the same as in (1.4), so we know that the corresponding Euler equation is given by (1.9). As the RA does not have any counterparty to trade with, we can immediately conclude that in equilibrium aggregate consumption must be the same as aggregate income in each period, hence $C_1 = Y_1 = 4$ and $C_2 = Y_2 = 4$. Plugging these values into the Euler equation, we see that

$$\frac{1}{C_1} = (1 + r^*)\frac{1}{C_2} \implies \frac{1}{Y_1} = (1 + r^*)\frac{1}{Y_2} \implies \frac{1}{4} = (1 + r^*)\frac{1}{4}$$

The only r^* that satisfies this equation is $r^* = 0$, which is the same as in the heterogeneous-agent economy in Example 1.2! We conclude that in the aggregate, both quantities (C_t and Y_t) as well as prices (r) are the same in the representative and the heterogeneous-agent economies. If we are only interested in aggregates, we can consequently model this economy using a representative agent. ■

Example 1.5 (Aggregation without borrowing). We now turn to aggregation if borrowing is not allowed, as in Example 1.3. We again assume that a representative agent (RA) solves (1.4) and has the same first-order conditions. Aggregate income is still given by $Y_1 = Y_2 = 4$. You'll notice that everything here is identical to Example 1.4, so we conclude that aggregate consumption is $C_1 = C_2 = 4$ and the equilibrium interest rate is given by $r^* = 0$.

However, recall that in the heterogeneous-agent economy we found that $r = -\frac{2}{3}$, so $r^* \neq r$ and this economy does *not* aggregate! ■

Aggregation usually fails if we in introduce some friction which prevents households from smoothing consumption or fully insuring against idiosyncratic risk, such as the borrowing limit imposed in Example 1.5. The presence of such frictions spawned a huge literature on how to solve heterogeneous-agent models using numerical methods towards the end of the 1980s, with seminal early contributions including Bewley (1977), Imrohoroğlu (1989), Huggett (1993), Aiyagari (1994) and Krusell and Smith (1998).

So far, we have only discussed heterogeneity on the household side. However, the same questions can be raised about the supply side of the economy, i.e., to firms. Do we lose important insights if we assume a single representative firm, as opposed to a distribution of firms which differ in size or productivity? There is a big literature investigating how frictions on the firm side (e.g., financial frictions such as access to credit) affect aggregate outcomes (see, for example, Khan and Thomas, 2008, 2013). We won't be concerned with production in this part of the course, so we'll illustrate this topic with a single example.

Example 1.6 (Firm aggregation). Consider a firm with a standard Cobb-Douglas production function, $Y = F(K, L) = K^\alpha L^{1-\alpha}$, which takes the interest rate r and wage rate w as given. The firm maximizes profits,

$$\Pi = K^\alpha L^{1-\alpha} - rK - wL \; ,$$

by choosing the optimal values of K and L. The first-order conditions state that the profit-maximising K and L must satisfy

$$r = \alpha \left(\frac{K}{L}\right)^{\alpha-1} \tag{1.24}$$

$$w = (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} \tag{1.25}$$

The above equations pin down the optimal *ratio* K/L but not K or L individually: if a firm optimally chooses some K and L, then λK and λL for some positive λ also satisfy (1.24) and (1.25)! This result follows from the fact that the Cobb-Douglas production function is *constant returns to scale*, i.e., it satisfies the property

$$F(\lambda K, \lambda L) = \lambda Y$$

for any value of λ.

We can now see why aggregation holds with an arbitrary number of firms. Imagine two firms, A and B, one with K_A and L_A such that K_A/L_A satisfies (1.24) and (1.25), and a second firm which chooses $K_B = 2 \cdot K_A$ and $L_B = 2 \cdot L_A$ instead, which also satisfies the first-order conditions. The first firm produces $F(K_A, L_A) = Y_A$, while the second firm's output is $F(2 \cdot K_A, 2 \cdot L_A) = 2 \cdot Y_A$.

Alternatively, instead of two firms, we could represent the production side of the economy by a single firm which employs $K = 3 \cdot K_A$ and $L = 3 \cdot L_A$ and thus produces $F(3 \cdot K_A, 3 \cdot L_A) = 3 \cdot Y_A$. This setup yields exactly the same output, uses the same amount of inputs and has the same equilibrium prices r and w! If we are not interested in the firm size distribution per se, we can therefore model a single representative firm without loss of generality! ■

1.4 Measures of inequality

In the preceding section, we established that the distribution of income across households can matter for aggregate outcomes. Moreover, we might be interested in questions about inequality in their own right, for example if we are devising policies to alleviate poverty. To this end, it is instructive to examine the empirical evidence that documents the extent of inequality in consumption, income, and wealth. Before we look at the data, however, we need to familiarise ourselves with the central measures used to quantify inequality.

1.4.1 Lorenz curve and Gini coefficient

Probably the most widely used way to characterize inequality is to report the Gini coefficient of some distribution, e.g., of income or wealth. Intuitively, the Gini represents how far this distribution deviates from perfect equality, so a Gini of 0 says that each person has the same income or wealth, whereas a Gini of 1 implies that everything is owned by one individual.[6]

Usually, we define the Gini coefficient in terms of the Lorenz curve, as shown in Figure 1.4. This graph is generated by ordering all individuals (or households) in terms of the variable we are interested in, starting with the most disadvantaged. On the y-axis we plot the cumulative share of resources owned by the bottom x of the population. In the example shown in the figure, the bottom 25% (the first quartile) jointly own 6%, while the first three quartiles together have 56%. The Gini coefficient can be computed using the size of the areas marked A and B in the figure using the formula

$$\mathcal{G} = \frac{A}{A+B} = 2A = 1 - 2B\,. \tag{1.26}$$

The two alternative expressions follow because $A + B = \frac{1}{2}$ since these areas cover exactly half of the 1×1 square. The two extreme cases with $\mathcal{G} = 0$ and $\mathcal{G} = 1$ are shown in Figure 1.5.

[6] In most cases, the Gini can take on values between 0 and 1. However, if the variable of interest can be negative (such as net worth, i.e., gross assets minus debt), the Gini can be larger than 1.

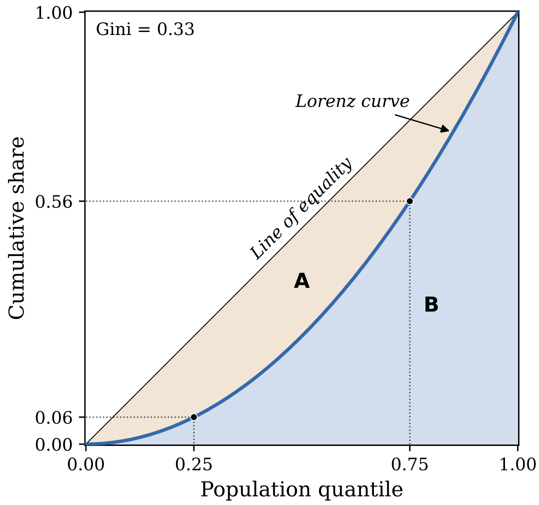


Figure 1.4: Lorenz curve and graphical representation of the Gini coefficient, which is given by $\mathcal{G} = A/(A+B) = 2A = 1 - 2B$. In this example, the lower quartile owns 6% of resources, while the first three quartiles own 56%.

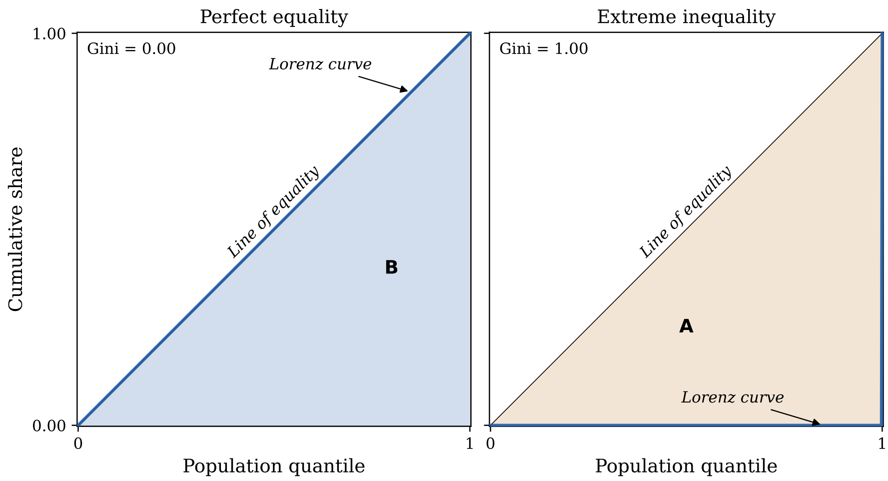


Figure 1.5: Lorenz curve and Gini for the extreme cases of "perfect" equality and inequality.

Be sure to plug the values for A and B into (1.26) to verify that the Gini evaluates to $\mathcal{G} = 0$ in the left case and to $\mathcal{G} = 1$ on the right!

To get a better feeling for these inequality measures, we next look at two simplified examples that nevertheless reflect the distribution of household income and wealth in the United States.

Example 1.7. Consider the following economy consisting of five households with income distributed as shown in Table 1.1.

Household	Income in $	Share	Cumulative share
1	15,750	3.0%	3.0%
2	35,650	6.7%	9.7%
3	58,950	11.1%	20.8%
4	96,790	18.2%	39.0%
5	324,090	61.0%	100.0%

Table 1.1: Hypothetical income distribution for economy of five households.

This closely matches the average family income by income quintile in the *Survey of Consumer Finances* for the United States in 2019.[7] Figure 1.6 show the Lorenz curve for this economy. The Gini is approximately 0.51, which comes reasonably close to the actual value for the US income distribution.

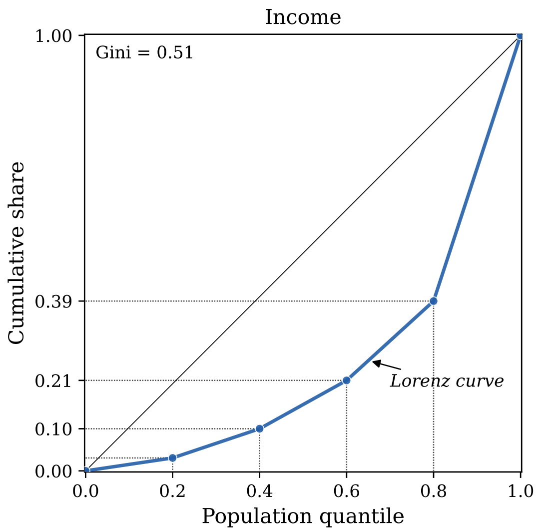


Figure 1.6: Lorenz curve and Gini for income distribution in example Example 1.7.

■

Example 1.8. Assume that the economy is populated by four households with wealth holdings shown in Table 1.2.

Household	Wealth in $	Share	Cumulative share
1	−13,630	−0.5%	−0.5%
2	58,180	1.9%	1.5%
3	236,280	7.9%	9.4%
4	2,706,290	90.6%	100.0%

Table 1.2: Hypothetical wealth distribution for economy of four households.

[7] See https://www.federalreserve.gov/econres/scf/dataviz/scf/chart/ for the original data and details.

This closely matches the average household net worth (gross assets minus debt) by quartile in the *Survey of Consumer Finances* for the United States in 2019. Figure 1.7 shows the Lorenz curve for this economy. The Gini is approximately 0.7, which is below the true value for the US as this simple economy does not adequately capture the wealth concentration at the very top. Moreover, because the lowest quartile has *negative* net

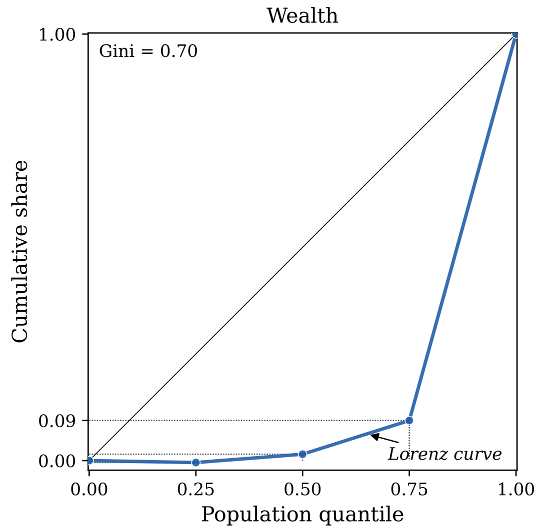


Figure 1.7: Lorenz curve and Gini for wealth distribution in example Example 1.8.

worth (i.e., more debt than assets), the Lorenz curve is initially decreasing and below zero! ■

1.4.2 Other measures

A distribution is a high-dimensional object, so there is no unique way to summarise it in a single number such as the Gini. There are a few additional measures of inequality used in the literature which we mention briefly. These measures differ in how sensitive they are to changes in specific parts of the distribution.

For example, the *variance of logs* is computed as the variance of an empirical distribution after taking the logarithm (the variance is a measure of dispersion around the mean). Unlike the Gini, the variance of logs is not very sensitive to changes at the top of the distribution because the logarithm compresses these values.

Other frequently-reported measures are the 90-10, 90-50 and 50-10 ratios. These measure the relative distance between two percentiles of a distribution. For example, the 90-10 ratio quantifies how much larger the 90^{th} percentile is compared to the 10^{th} percentile. If the 90-10 ratio for income is 5, this means that a household at the 90^{th} percentile of the income distribution earns five times as much as the household at the 10^{th} percentile. These three ratios together allow us to get a more disaggregated picture of inequality, whereas the Gini does not. For example, if the 90-50 ratio increased while the 50-10 remained the same, we can conclude that inequality remain roughly unchanged in the bottom half of the distribution whereas the top decile did better than the median.

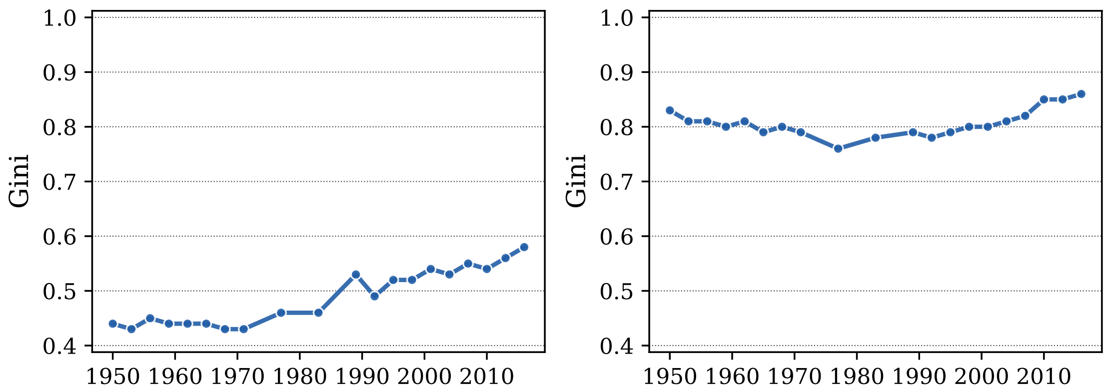
1.5 Inequality in the data: some facts for macroeconomists

To motivate our discussion of heterogeneity in macroeconomics, we need to go beyond aggregate time series (for GDP, aggregate consumption, etc.) and use *micro data* to document differences across households along various dimensions such as consumption, income, or wealth. Appendix 1.1 lists some of the most important publicly available socio-economic data sets for the US and UK which can be used for such an exercise. The next section present some of the findings obtained from these data.

1.5.1 Inequality in the United States

Income and wealth inequality

We start by looking at the evolution of the income and wealth distribution in the US over time. Figure 1.8 shows how the Gini coefficient for each of these variables changed since the 1950s. The data comes from the *Survey of Consumer Finances* (SCF) and its precursors and was compiled by Kuhn, Schularick, and Steins (2020). You might have read newspaper reports about the earnings polarization and the increase in income inequality over the past decades; this is documented in Figure 1.8 panel (a), which shows a rise of the income Gini from 0.43 in 1971 to 0.58 in 2016. Panel (b) shows the wealth Gini, which was u-shaped over the post-war period but mostly hovered around a value of 0.8. As you can see, wealth is substantially more unequally distributed than income!

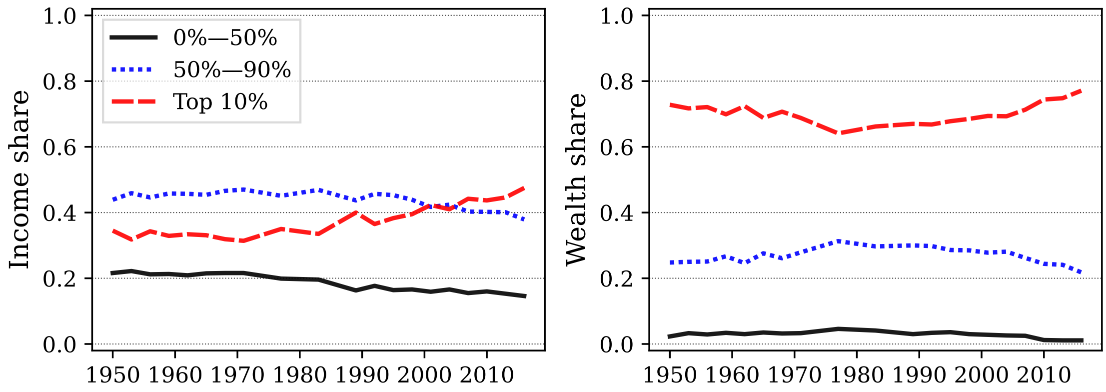


(a) Income

(b) Wealth

Figure 1.8: Gini for gross household income (including transfers) and household net worth in the US, 1950–2016. Data source: Kuhn, Schularick, and Steins (2020, Table E.5)

One disadvantage of the Gini is that it does not make apparent how exactly income or wealth is distributed, other than that the distribution is more or less equal. For example, we could have economies that are unequal because the top earners are disproportionately rich, or because the wealth-poor have high levels of debt. These two economies could



(a) Income

(b) Wealth

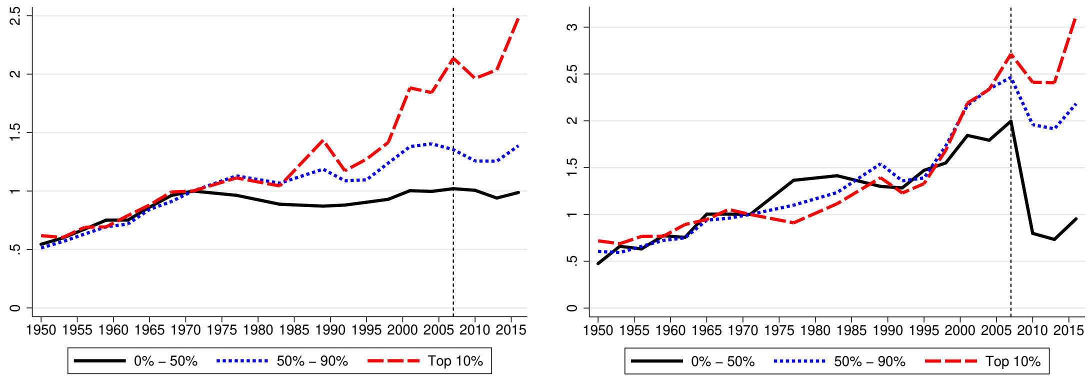
Figure 1.9: Shares of income and wealth in the US, 1950–2016. The left panel shows the shares of aggregate income going to the bottom 50%, the middle class (50%–90%) and the top 10% of income earners. The right panel shows share of aggregate wealth holdings, and the population is now partitioned by wealth, i.e., the bottom group refers to the 50% of households with the lowest wealth. Data source: Kuhn, Schularick, and Steins (2020, Table E.4)

potentially give rise to the same Gini coefficient. Consequently, we often want to look at more disaggregated statistics such as the income and wealth shares held by various groups, as illustrated in Figure 1.9. Panel (a) depicts the evolution of income shares in the US over the last decades and highlights the increasing concentration at the top: the 10% income richest were able to increase their share of aggregate income from 35% to almost 48% in 2016 at the expense of both the middle class and the bottom 50%. Panel (b) plots the corresponding time series for wealth: in line with our observations for the Gini, we again see that wealth is much more concentrated, with the richest 10% of households owning around 75% of all wealth in the US.

Another approach to assess how different groups are doing over time is to look at their income or wealth growth. This is accomplished in Figure 1.10. In each panel, the real income or wealth of each group is normalised to a value of one in 1971, so we can read off the growth that each group experienced. For example, panel (a) shows that the bottom 50% saw no income growth between 1971 and 2007, since their normalised income was approximately one at both points in time. On the other hand, the top 10% saw their real income more than double in the same period!

Turning to wealth accumulation, from Figure 1.10 panel (b) we see that all groups experienced similar growth until the Great Recession of 2007, but their wealth holdings fared very differently thereafter: the bottom 50% saw a rapid decline that wiped out almost all gains over the past decades! This highlights the importance of taking into account household heterogeneity when quantifying the cost of business cycles.

What caused these large differences in wealth trajectories after the Great Recession? Kuhn, Schularick, and Steins (2020) and others argue that the underlying reason are differences in portfolios along the wealth distribution. This is illustrated in Figure 1.11



(a) Income growth

(b) Wealth growth

Figure 1.10: Income and wealth growth for the bottom 50%, the middle class (50%–90%) and the top 10% of the wealth distribution. All time series are normalised to one in 1971. The dashed vertical line in 2007 shows the Great Recession. Source: Kuhn, Schularick, and Steins (2020, Figure 12)

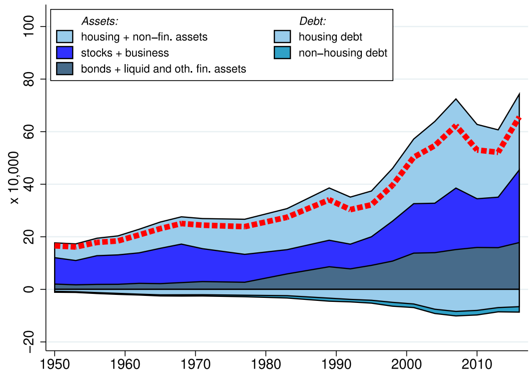
which plots the average portfolios for selected groups of households. It is evident that there is substantial portfolio heterogeneity along the wealth distribution: for the bottom 50% in panel (b), the most important asset is housing (which is financed by a mortgage), whereas the top 10% in panel (d) have substantial holdings of equities and businesses. Because house prices collapsed after 2007, the bottom 50% with their leveraged real estate suffered disproportionately more than other groups.

Consumption and leisure inequality

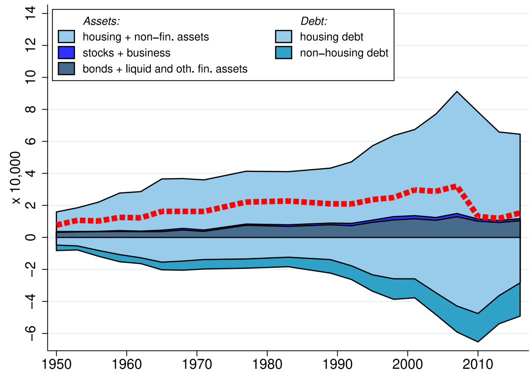
So far, we discussed the distribution of income and wealth. However, economists for the most part believe that welfare depends on consumption and leisure but not directly on income or wealth — recall that our utility function is usually written as $u(c)$ or $u(c, \ell)$.

Consumption differs from income for a variety of reasons: households save part of their income, they pay taxes and receive government transfers, or they may receive transfers from family members. Because households can to some extent insure themselves against adverse income shocks via savings (or credit), and since the tax and transfer system redistributes to income-poor households, consumption ends up being more equally distribution than income or wealth. However, there is some disagreement about the trends in consumption inequality: earlier studies such as Krueger and Perri (2006) found that consumption inequality did not increase nearly as much as income inequality over the last decade. More recent literature, on the other hand, concludes that consumption and income inequality track each other closely (see Attanasio and Pistaferri (2016) for a survey).

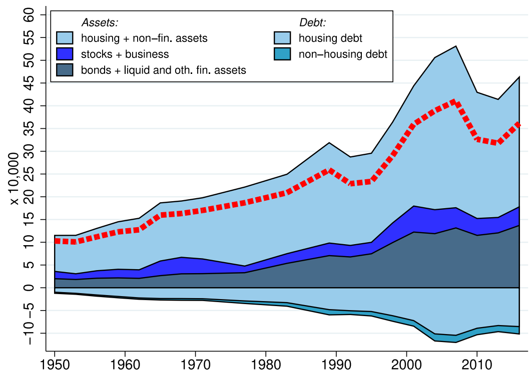
As an example, Figure 1.12 documents this increase in food consumption inequality by plotting the log difference between the 90$^{\text{th}}$ and 10$^{\text{th}}$ percentile of the consumption distribution in the US over time. Looking at the red line, we see that the difference was



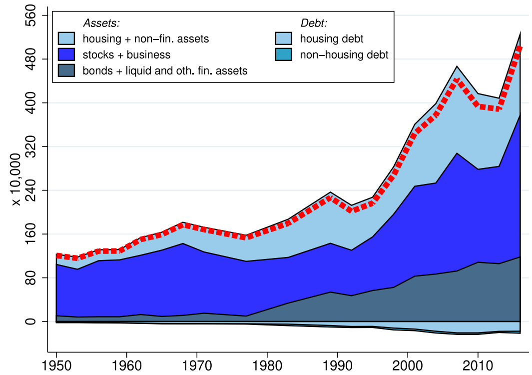
(a) All households



(b) Bottom 50%



(c) 50%–90%



(d) Top 10%

Figure 1.11: US household portfolios of different groups along the wealth distribution. Assets are shown as shaded areas above the zero line while liabilities are displayed as negative values. The red dashed line depicts the net worth (assets minus debt). Panel (a) shows the average portfolio across all households, panels (b), (c) and (d) the portfolios for the bottom 50%, the middle class and the richest 10%, respectively. All values are reported in 10,000 USD at 2016 prices. Source: Kuhn, Schularick, and Steins (2020, Figure 14)

approximately 1.2 log points in 1977, so that

$$\log c_{90} - \log c_{10} \approx 1.2 \tag{1.27}$$

which means that the consumption expenditures of a household at the 90$^{\text{th}}$ percentile were more than 3 times higher than those of a household at the 10$^{\text{th}}$ percentile ($c_{90}/c_{10} \approx \exp(1.2) \approx 3.3$).[8] By 2012, this wedge increased to approximately 1.6 log points or a factor of 5! Of course, spending on food does not fully capture the actual quantity of

[8] Economists like to use log differences because these approximate the relative difference of two quantities x and y, i.e., $\frac{x-y}{x} \approx \log x - \log y$. However, this approximation only works if x and y are close, and it fails miserably in cases such as (1.27) where the true difference is 230%, not 120% as suggested by the approximation.

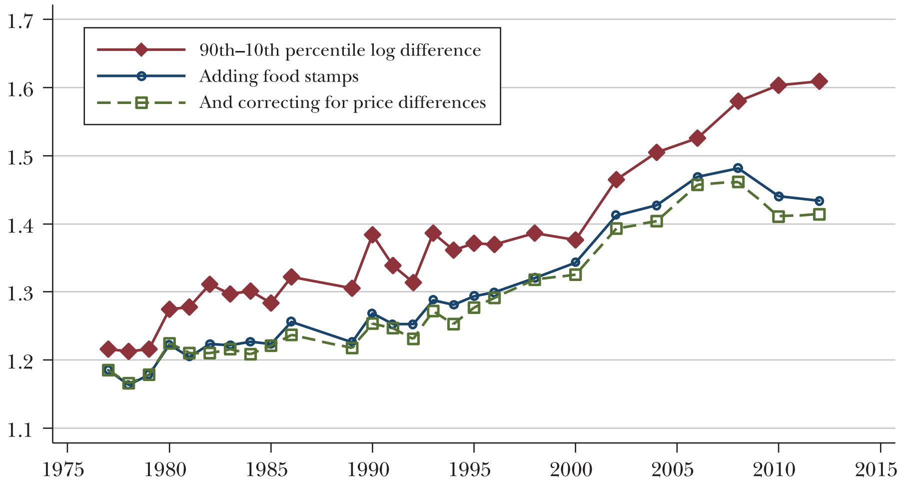


Figure 1.12: Difference between the 90$^{\text{th}}$ and the 10$^{\text{th}}$ percentiles of distribution of the logarithm of food consumption, 1977–2012. Source: Attanasio and Pistaferri (2016, Figure 2), based on PSID data.

food consumed: for example, eligible households in the US receive food stamps which do not show up in spending, so food expenditure alone would overstate inequality in food consumption. Nevertheless, even after accounting for such in-kind transfers (see the blue line in Figure 1.12), the increase in consumption inequality remains.

Another way to quantify consumption inequality is to look at at ownership rates of durable goods such as cars or home appliances, as shown in Figure 1.13. As the panels illustrate, ownership rates of durable goods across the bottom and top income deciles converged for some categories, but remain quite different for others such as cars. For example, among the 10% of highest income earners, more than 90% report owning car, while this is true for only 70% of the bottom 10%.

Turning to leisure, Figure 1.14 shows the trends in leisure time by gender and educational attainment. In principle, we might think that households can choose between two allocations: high consumption financed by supplying labour, and low consumption that comes with additional leisure. This is one possible interpretation of the data in Figure 1.14 which shows that men who did not complete high school (and are thus more likely to have low income) enjoy 10–15 more hours of leisure per week compared to college-educated men. However, this increased leisure might not be entirely voluntary, as this group most likely lacks the job market opportunities of the college educated! Attanasio and Pistaferri (2016) point out that this leisure gap is smaller when the sample includes only the employed, so part of this difference is attributable to differences in (in)voluntary unemployment or retirement across education groups.

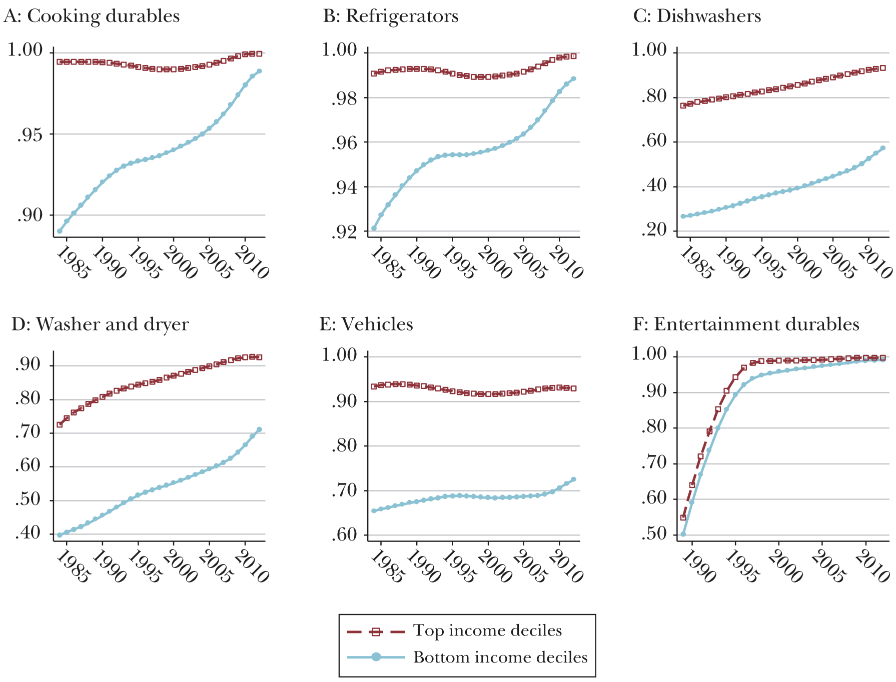


Figure 1.13: Ownership rates for selected durables for top and bottom after-tax income deciles. Source: Attanasio and Pistaferri (2016, Figure 3), based on CEX.

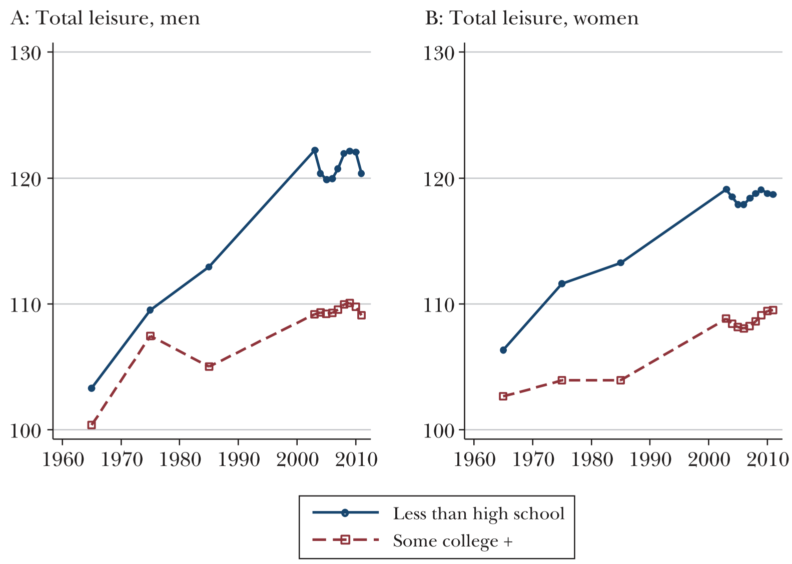


Figure 1.14: Total leisure hours per week, defined as the sum of social activities, active and passive leisure, and time devoted to personal care (which includes sleeping). Source: Attanasio and Pistaferri (2016, Figure 4), based on US time use data.

1.5.2 Inequality in the United Kingdom

As documented in Belfield et al. (2017), income inequality has been on the rise in the UK over the last decades as well, even though the UK and the US have seen somewhat different trajectories since the 1990s. As shown in Figure 1.15, there was a pronounced increase in the 1970s and 1980s: the income Gini increased from 0.26 in 1980 to 0.34 in 1990. This increase was in part due to rising wage inequality because of skill-biased technical change (i.e., some occupations benefiting from technological progress more than others), but also due to weaker trade unions and regressive changes to the tax and transfer system.

Thereafter, income inequality remained broadly constant, except for the upper tail of the income distribution where top earners were able to increase their share of income. However, judging by the 90-10 percentile ratio, income inequality actually decreased for large parts of the income distribution (see the 90-10 ratio in Figure 1.15), contrary to what we observed in the US in that time period. One reason for this development were changes to the tax and benefit system introduced in the later parts of the 1990s, in particular in the form of cash transfers which were targeted towards the poorer half of working households.

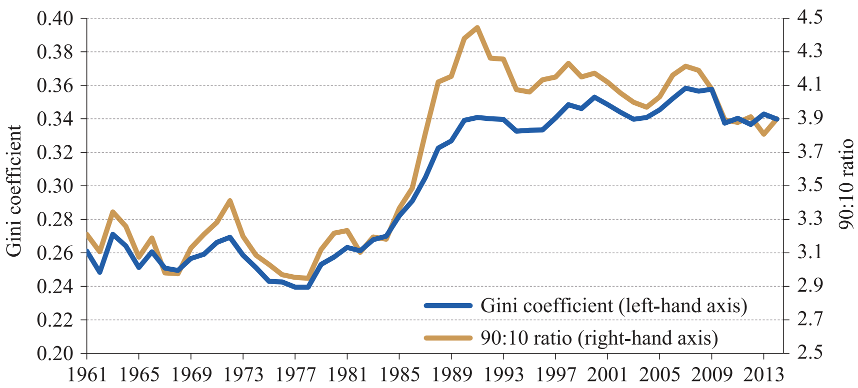


Figure 1.15: The Gini coefficient and the 90-10 ratio of net household income (adjusted for household size) in Great Britain, 1961–2014. Source: Belfield et al. (2017, Figure 2)

It is insightful to look at how income inequality is mitigated via the tax and transfer system, which is shown in Figure 1.16 for UK households. Gross income (black line) is the most unequally distributed, whereas this inequality is dampened by the progressive tax system (red line) and public transfers (pink line). Finally, disposable income (green line), which might additionally include intra-family transfers, shows the lowest levels of inequality.

Blundell and Etheridge (2010) provide numerous additional graphs documenting various dimensions of inequality in the UK, including consumption inequality.

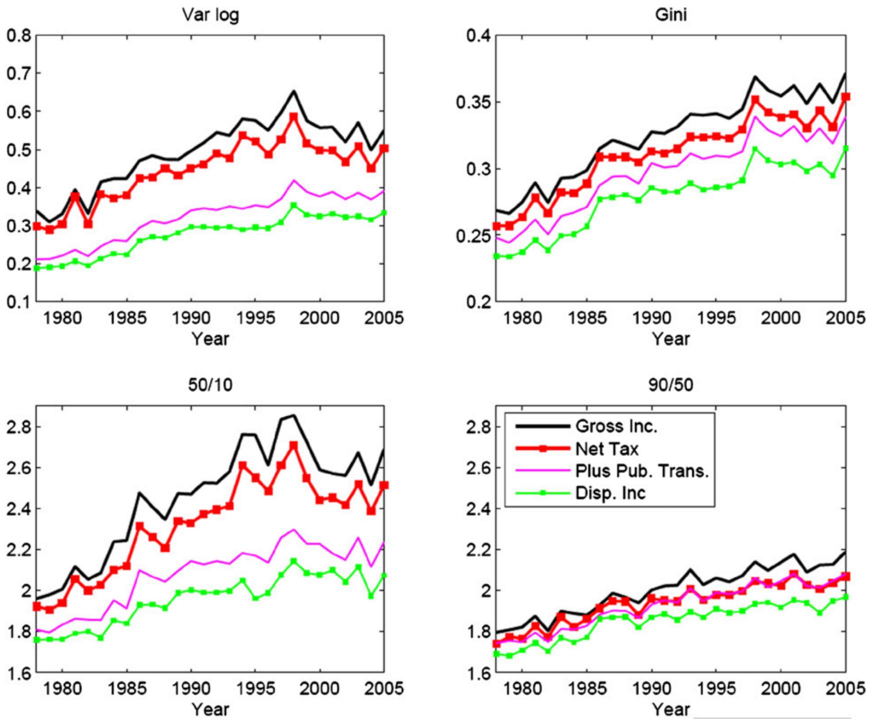


Figure 1.16: Change in inequality when moving from gross income to disposable income. Source: Blundell and Etheridge (2010, Figure 4.4), based on FES data

1.6 Main takeaways

In terms of modelling, we introduced three new concepts in this unit:

1. We studied how to solve simple general-equilibrium models with *heterogeneous agents* where heterogeneity was restricted to two types of households.

2. We introduced the concept of constrained optimisation in the form of household problems with *borrowing constraints*. Solving such problems adds additional complications since we need to check whether a constraint is binding in equilibrium.

3. We discussed the concept of *aggregation*, i.e., whether an economy with heterogeneous agents yields the same aggregate quantities and prices as a representative-agent economy.

   We found that borrowing constraints are one reason why aggregation fails.

Additionally, we covered empirical evidence on inequality in the UK and US, which can be summarised as follows:

1. Ranked in terms of inequality, wealth is the most unequally distributed resource, followed by income and consumption.

2. Redistributive taxes and transfers mitigate some of the income inequality, so disposable income is more equally distributed than pre-tax income.
3. Income inequality has increased over the last decades, and more strongly in the US than the UK. There is some evidence that this was accompanied by an increase in consumption inequality.
4. We use multiple statistics to quantify inequality: the Gini coefficient, the variance of logs, or the 90-10, 90-50 or 50-10 percentile ratios. Each of these highlights inequality in some part of the distribution, but may be less sensitive to other parts.

Appendix 1.1: Micro data for the US and UK

To study inequality, we need to go beyond aggregate times series (such as GDP, aggregate consumption, etc.) and use data on individuals or households. These come in two broad varieties: panel (or longitudinal) data follow the same respondents for a prolonged period of time (often decades), while (repeated) cross-sections draw a new sample of individuals or households every time a survey is administered. A rotating panel is a hybrid variant which interviews the same respondent for a limited number of times (say 2–4 times in consecutive months or quarters), but thereafter brings in a new set of individuals or households.

US data sets

Some of the most important publicly available data sets to study inequality (and other issues) in the US are the following:

1. The *Current Population Survey (CPS)* is more-or-less a repeated cross-section of households collected at monthly frequencies by the US Census Bureau (strictly speaking, the CPS is a very short panel because each household is interviewed twice). It is frequently used to compute labour-market statistics such as the US unemployment rate.[9]

2. The *Panel Study of Income Dynamics (PSID)* was started in 1968 and is the longest-running longitudinal household survey in the world. Initially, it interviewed 5,000 families every year, but switched to a biennial frequency in 1997. It is administered by the University of Michigan.[10]

3. The *Health and Retirement Study (HRS)* is a panel data set of predominantly elderly households (aged 50 or older). In addition to income and wealth, it contains a multitude of variables which are particularly relevant for studying old age, such as health and medical information. It is administered by the University of Michigan.[11]

4. The *Survey of Consumer Finances (SCF)* is a survey administered as a repeated cross-section every three years. It attempts to oversample rich household so that it is better-suited to answer questions of wealth inequality than most other data sets. The survey is run by the US Federal Reserve.[12]

5. The *Consumption Expenditure Survey (CEX)* collects data on households' consumption of non-durables and selected durables (such as cars) in the US since 1980. Households are interviewed for four consecutive quarters, so this survey is administered as a rotating mini-panel. It is run by the US Bureau of Labor Statistics.[13]

[9] https://www.census.gov/programs-surveys/cps.html
[10] https://psidonline.isr.umich.edu/
[11] https://hrs.isr.umich.edu/about
[12] https://www.federalreserve.gov/econres/scfindex.htm
[13] https://www.bls.gov/cex/

UK data sets

Similar data sets exist for the United Kingdom, for example:

1. The *British Household Panel Survey (BHPS)* is a longitudinal survey that interviewed households annually between 1991–2008.[14]

2. The *Understanding Society* study, or the United Kingdom Household Longitudinal Study (UKHLS), is a longitudinal survey that replaced the BHPS in 2009.[15]

3. The *Labour Force Survey (LFS)* collects data on labour market and employment outcomes. It started as a biennial survey in 1973, but has by now transitioned to a quarterly frequency which contains a short rotating panel of 5 quarters.[16]

4. The *Family Resources Survey (FRS)* is a repeated cross-section administered annually by the UK government and contains about 20,000 households each financial year. It surveys all income received by households as well the amount of direct taxes paid.[17]

5. The *Living Costs and Food Survey (LCF)* collects data on spending patterns and the cost living in the UK. From 2001–2008, this data was collected as part of the *Expenditure and Food Survey (EFS)*, which itself superseded the *Family Expenditure Survey (FES)* that collected such data from 1957–2001.[18]

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Exercises

Exercise 1.1 (General equilibrium with discounting). Consider the setting in Example 1.1, but now assume that the household discounts period-2 utility with $\beta \in (0, 1)$. The household problem is given by

$$
\begin{aligned}
&\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \beta \log(c_2) \qquad \text{(E.1)} \\
&\text{s.t.} \quad c_1 + a_2 = y_1 \\
&\qquad\quad c_2 = (1+r)a_2 + y_2
\end{aligned}
$$

Assume that the household can freely borrow or save between the two periods.

(a) Consolidate the per-period budget constraints into a single present-value lifetime budget constraint.

(b) Set up the Lagrangian for this problem and derive the first-order conditions. Use these to obtain the household's Euler equation.

(c) Use the Euler equation and the budget constraint to solve for optimal consumption (c_1, c_2) as a function of the interest rate, income and parameters.

(d) As in Example 1.2, assume that the economy is populated by two households, A and B, with the following income streams:

$$
\begin{aligned}
y_1^A &= 3, \; y_2^A = 1 \\
y_1^B &= 1, \; y_2^B = 3
\end{aligned}
$$

Use the optimal consumption rules you found to determine the equilibrium interest rate as function of income and parameters.

(e) Assume that the discount factor is given by $\beta = 0.8$. Compute the equilibrium interest rate!

(f) Comment on the magnitude of equilibrium r compared to what we found in Example 1.2. Explain the effect of lowering β from 1 to 0.8!

(g) Extend the graphs in Figure 1.2 to include the equilibrium for $\beta = 0.8$.

Exercise 1.2 (Representative-agent with CRRA preferences). Consider the following two-period representative-agent problem,

$$
\max_{c_1,\, c_2,\, a_2} \quad u(c_1) + \beta u(c_2) \tag{E.2}
$$

$$
\text{s.t.} \quad c_1 + a_2 = y_1 \tag{E.3}
$$

$$
c_2 = (1+r)a_2 + y_2 \tag{E.4}
$$

where $u(\bullet)$ is the CRRA utility function

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

and $\gamma \neq 1$ is a parameter governing the curvature. The household receives income y_t in period t and can choose to save or borrow in the first period.

(a) State the Lagrangian for this maximisation problem. You can either eliminate all choice variables other than a_2 or keep all three choice variables and include the constraints (E.3) and (E.4).

(b) Derive the optimality condition for this problem. Irrespective of how you chose to set up the Lagrangian, you should eventually end up with an Euler equation.

(c) Find an expression for the equilibrium interest rate as a function of income and parameters. To do this, recall that in this representative-agent economy, there is no way to save or borrow in equilibrium, and therefore the representative agent has to consume its endowment in each period. Find the interest rate that supports this allocation!

(d) Intuitively explain how the equilibrium interest rate depends on the discount factor β and the ratio of second- to first-period income, y_2/y_1.

(e) Assume that $\beta = \gamma = 1$ and consider three different scenarios for income:

Scenario	y_1	y_2	r
A	3	1	?
B	1	3	?
C	2	2	?

Compute the equilibrium interest rate for each scenario and plot the equilibria in a graph with c_1 and on the x-axis and c_2 on the y-axis. Include both the budget lines and indifference curves, and clearly label all elements!

Exercise 1.3 (General equilibrium with limited borrowing). Consider the following household problem:

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \log(c_2) \tag{E.5}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = (1+r)a_2 + y_2$$

$$a_2 \geq -b \tag{E.6}$$

The household is able to borrow funds in period 1, but only up to an amount b, the borrowing limit, as shown in the constraint (E.6).

(a) Write down the Lagrangian for problem (E.5), including both per-period budget constraints and the borrowing constraint (E.6).

(b) Take the first-order conditions with respect to c_1, c_2 and a_2. Consolidate all three conditions into a single Euler equation.

(c) Assume there are two households, A and B, in this economy with income given by

$$y_1^A = 3,\ y_2^A = 1$$
$$y_1^B = 1,\ y_2^B = 3$$

Furthermore, let the borrowing limit be $b = 1$.

Use a "guess and verify" approach to find the equilibrium: you suspect that both households can perfectly smooth consumption like in Example 1.2, so you conjecture that B will borrow up to the borrowing limit, and A will supply the necessary savings. Use this conjecture to determine the equilibrium interest rate.

(d) Plot the equilibrium allocation for A and B in a graph with c_1 and on the x-axis and c_2 on the y-axis. Include both the budget lines and indifference curves, and clearly label all elements (create a separate graph for each household).

Exercise 1.4 (Aggregation with heterogeneous preferences). Consider an economy with three different households indexed by $i = A, B, C$ which solve the following consumption-savings problem:

$$\begin{aligned} \max_{c_1,\, c_2,\, a_2} \quad & u_i(c_1) + \beta u_i(c_2) \\ \text{s.t.} \quad & c_1 + a_2 = a_1 + y_1 \\ & c_2 = (1+r)a_2 + y_2 \end{aligned} \tag{E.7}$$

Each household is assumed to receive the same endowment (y_1, y_2) and can borrow freely. Assume that the per-period utility depends on the household type and is given as follows:

$$\begin{aligned} u_A(c) &= \frac{c^{1-\gamma}}{1-\gamma} \\ u_B(c) &= \frac{c^{1-\gamma}}{1-\gamma} + 100 \\ u_C(c) &= 2 \cdot \frac{c^{1-\gamma}}{1-\gamma} \end{aligned}$$

(a) Derive the Euler equation for each household.

(b) Does this economy aggregate, i.e., can it be modelled as a single representative household? Compare the optimality condition you found above to find the answer!

Exercise 1.5 (Properties of inequality measures). Consider the following income distribution:

Household	Income in £
1	10,000
2	20,000
3	30,000
4	40,000
5	100,000

(a) Compute the cumulative income shares along the income distribution, i.e., the share of aggregate income received by the first n households for $n = 1, 2, \ldots, 5$.

(b) Use this data to plot the Lorenz curve for this economy.

(c) Assume that the income of each household is multiplied by a factor of 10. What happens to income inequality in this economy?

Exercise 1.6 (Ranking distributions). Consider four different economies with the following distributions of income:

Household	Economy			
	1	2	3	4
1	0	20	0	10
2	0	40	0	20
3	200	60	0	30
4	200	80	100	40

(a) Plot the Lorenz curves for all four economies.

(b) Rank these economies in terms of their inequality. You can infer this directly from the table or by looking at the Lorenz curves.

Exercise 1.7 (Inequality with taxes and transfers). Consider the following distribution of gross income:

Household	Gross income
1	5
2	10
3	15
4	50

Table E.1: Distribution of gross income

(a) Compute the cumulate shares of gross income.

(b) Assume that the government introduces a progressive tax schedule given by

$$T(y) = y - y^{1-\tau}$$

where y is pre-tax income, $T(y)$ is the tax liability, and $\tilde{y} = y - T(y)$ is the after-tax income. Let $\tau = 0.18$, which was found to be a good approximation to the progressivity of the US tax system.[19]

Apply this tax schedule to the gross income distribution tabulated in Table E.1 and compute the (cumulative) shares of after-tax income!

(c) Assume that a pandemic hits the original economy (without taxes), and the government distributes one unit of income to each household as an unconditional cash transfer.

Compute the post-transfer (cumulative) income shares for the distribution from Table E.1.

(d) Using your calculations from above, plot the Lorenz curves for all three scenarios. How do progressive taxes and cash transfers affect inequality in this economy?

[19]See Jonathan Heathcote, Kjetil Storesletten, and Giovanni L. Violante (2017). "Optimal Tax Progressivity: An Analytical Framework". In: *The Quarterly Journal of Economics* 132.4, pp. 1693–1754.

2 Consumption over the life cycle

2.1 Introduction

In this unit, we explore one simple reason for heterogeneity in household decisions: differences in age. The so-called *life cycle model* of consumption and savings explains why households make distinct decisions at different points in time, for example because they face age-dependent income (or no income in retirement).

Before studying the full model, we first examine how households adapt their intertemporal consumption choices in response to interest rate changes in a two-period setting. We decompose such responses into an income, wealth and substitution effect, and the latter will depend on the *elasticity of intertemporal substitution* which nicely ties into CRRA preferences.

Once this ground work is done, the two-period model can be naturally extended to many periods. The predictions of such a richer model can then be compared to age profiles of consumption and savings observed in the data.

2.2 Income, substitution and wealth effects

2.2.1 Log preferences

Let's begin with our standard example, the two-period consumption-savings problem with log preferences. For now, we additionally assume that the household only receives income in the first period. The maximisation problem is as follows:

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \beta \log(c_2) \tag{2.1}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1 \tag{2.2}$$

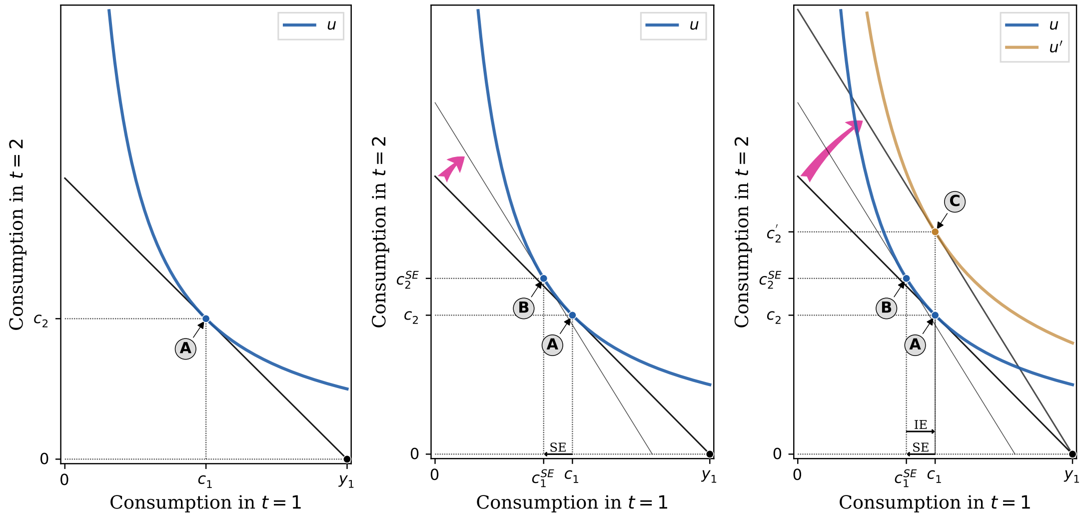
$$c_2 = (1+r)a_2 \tag{2.3}$$

We can consolidate the per-period budged constraints (2.2) and (2.3) to form the present-value lifetime budget constraint (LTBC)

$$c_1 + \frac{c_2}{1+r} = y_1\,, \tag{2.4}$$

which states that discounted lifetime consumptions is equal to the period-1 endowment y_1. We derived the Euler for this problem in the previous unit, which is given by

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2}\,. \tag{2.5}$$



(a) Initial allocation **(b)** Substitution effect **(c)** Total effect

Figure 2.1: Substitution and income effects of an increase in r for a lender with log preferences and no second-period income. Point Ⓐ depicts the optimum at the initial interest rate located on the associated indifference curve with utility u. Ⓑ shows the allocation at the new interest rate which yields the same utility level. Ⓒ is the new allocation once the total effect is taken into account, with the yellow line showing the new indifference curve with utility u'.

Solving for $c_2 = \beta(1+r)c_1$ and substituting into the lifetime budget constraint, we find the optimal consumption level in period 1:

$$c_1 = \frac{1}{1+\beta} y_1 \tag{2.6}$$

Plugging this back into the Euler equation, optimal consumption in period 2 is given by

$$c_2 = \frac{\beta}{1+\beta}(1+r)y_1 \ . \tag{2.7}$$

For example, if we set $\beta = 1$ and $r = 0$, (2.6) and (2.7) imply that the household consumes exactly half of its initial endowment in each period, as expected.

We are now in a position to ask the following question: how does the household respond to changes in r? From (2.6) we see that consumption in the first period does not respond at all as it is not a function of r! This might not be particularly surprising if you recall from your earlier studies that the substitution and income effects exactly cancel out with log preferences.

Figure 2.1 shows the intuition behind the result: if the interest rate increases, the relative price of period-1 consumption increases — recall that the price of period-1 consumption in terms for period-2 consumption is $(1+r)$. The resulting *substitution effect (SE)* is defined as the change in demand when relative prices move while keeping the utility level constant. This is illustrated in panel (b) by tilting the budget line along the indifference curve until its slope is $-(1+r')$, reflecting the new interest rate r'. As the slope changes, the cheaper c_2 is substituted for c_1. First-period consumption consequently moves from c_1 to c_1^{SE}, and the consumption bundle moves from **A** to **B**.[1]

That, however, is not the end of the story: since the household is a lender, it earns higher interest income on its savings than before. Because c_1 and c_2 are normal goods, demand for both increases compared to point **B**. This is called the *income effect (IE)*. Depending on preferences and whether the household is a lender (saver) or a borrower, the overall effect on c_1 might be ambiguous. However, in the case of a lender with log preferences and no period-2 income, the SE and IE exactly cancel and c_1 remains unchanged, as shown in panel (c) at point **C**. Moreover, because in this example the household is a lender, an increase in the interest rate makes it unambiguously better off, as shown by the new indifference curve.

You might wonder why we eliminated second-period income in the preceding discussion. It turns out that the exactly offsetting substitution and income effects only occur if wealth itself is not affected by the price change. However, if lifetime wealth is distributed across multiple periods, its present value will respond to changes in the interest rate. To illustrate, let's return to our standard version of the two-period model with income received in both periods.

Example 2.1 (Wealth effect with log preferences). Instead of a fixed endowment in the first period as in (2.1), the household now receives income in both periods and solves

$$\max_{c_1,\, c_2,\, a_2} \quad \log(c_1) + \beta \log(c_2) \tag{2.8}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1 \tag{2.9}$$

$$c_2 = (1+r)a_2 + y_2 \tag{2.10}$$

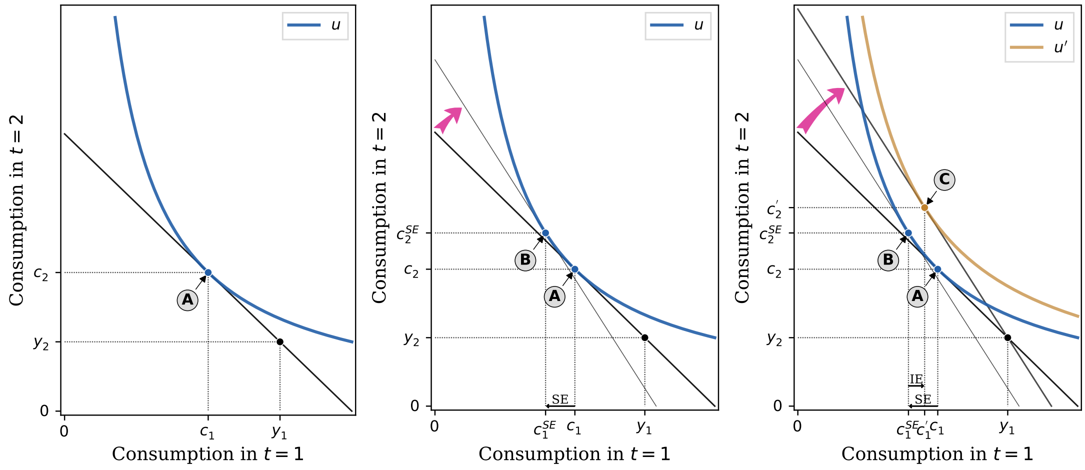
In the present-value lifetime budget constraint, lifetime income on the right-hand side now clearly depends on the interest rate:

$$\underbrace{c_1 + \frac{c_2}{1+r}}_{\text{PV of lifetime cons.}} = \underbrace{y_1 + \frac{y_2}{1+r}}_{\text{PV of lifetime inc.}} . \tag{2.11}$$

The Euler equation is unchanged from (2.5). Substituting for c_2 in the lifetime budget constraint and solving for c_1, we find that

$$c_1 = \frac{1}{1+\beta}\left[y_1 + \frac{y_2}{1+r}\right] \tag{2.12}$$

[1] The hypothetical demand schedule which represents demand as a function of relative prices at a constant utility level is called *Hicksian demand* or *compensated demand*. As the word "compensated" implies, the household is assumed to have been given the exact amount of resources required to attain the fixed utility level. Because utility is unobserved and the household might not actually have enough resources to purchase the consumption bundle, this demand function is "hypothetical."



(a) Initial allocation **(b)** Substitution effect **(c)** Total effect

Figure 2.2: Substitution and income effects of an increase in r for a lender with log preferences and income in both periods. Point (A) depicts the optimum at the initial interest rate located on the associated indifference curve with utility u. (B) shows the allocation at the new interest rate which yields the same utility level. (C) is the new allocation once the total effect is taken into account, with the yellow line showing the new indifference curve with utility u'.

while in the second period, consumption is

$$c_2 = \frac{\beta}{1+\beta}\Big[(1+r)y_1 + y_2\Big] \tag{2.13}$$

Looking at (2.12), we see that period-1 consumption is now unambiguously *decreasing* in the interest rate due to the *wealth effect*. A higher interest rate decreases the present value of period-2 income, an effect that was absent earlier. Previously we found that c_1 remained unchanged, so now it must be the case that the household reduces demand for all normal goods, including c_1. This situation is illustrated in Figure 2.2. Panels (a) and (b) are more-or-less unchanged from Figure 2.1, but in panel (c) period-1 consumption at the new allocation is lower than its initial value.[2]

To summarise, we tabulate all three effects of an increase in r for a lender:

[2]There does not seem to be any agreement on the correct terminology to describe this scenario. Standard microeconomics textbooks such Jehle and Reny (2011) define the income effect as the *residual* after the substitution effect is accounted for, whereas Jappelli and Pistaferri (2017) explicitly mention the wealth effect as the third term when decomposing interest rate changes.

Decomposition	$\partial c_1/\partial r$	
Substitution effect	< 0	
Income effect	> 0	
Wealth effect	≤ 0	Depends on timing of income
Total effect	?	

Table 2.1: Decomposition of change in lender's period-1 consumption following an increase in r.

Because the individual effects have opposite signs, the total effect is indeterminate at this level of generality and will depend on the exact numerical values used to solve the household problem (for example, Figure 2.2 was created using $\beta = 1$, $y_1 = 3$, $y_2 = 1$, $r = 0$ and $r' = 0.6$). ∎

2.2.2 CRRA preferences and the elasticity of intertemporal substitution

What determines the magnitude of the substitution effect between consumption in periods 1 and 2 as the interest rate changes? Looking at the above figures, one would conjecture that the propensity to shift consumption should be related to the curvature of the indifference curve. With log preferences, we have limited scope to control this curvature, and in fact we will see below that with this parametrisation, the elasticity of intertemporal substitution is hardwired to be exactly one. To gain more flexibility, we return to the more general class of CRRA preferences. Log preferences are a special case within this class which arises if the coefficient of relative risk-aversion is set to one.

Example 2.2 (Consumption growth with CRRA preferences). Consider the following problem of a household endowed with CRRA preferences who is allowed to save or borrow between periods 1 and 2:

$$
\begin{aligned}
&\max_{c_1,\, c_2,\, a_2} \; \frac{c^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma} \\
\text{s.t.} \quad & c_1 + a_2 = y_1 \\
& c_2 = (1+r)a_2 + y_2
\end{aligned}
$$

The present-value life-time budget constraint is the same as in (2.11). We set up the Lagrangian as usual,

$$
\mathcal{L} = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma} + \lambda \left[y_1 + \frac{y_2}{1+r} - c_1 - \frac{c_2}{1+r} \right]
$$

where $\lambda \geq 0$ is the Lagrange multiplier for the lifetime budget constraint. The first-order conditions with respect to c_1 and c_2 are

$$
\frac{\partial \mathcal{L}}{\partial c_1} = c_1^{-\gamma} - \lambda = 0 \tag{2.14}
$$

$$
\frac{\partial \mathcal{L}}{\partial c_2} = \beta c_2^{-\gamma} - \lambda \frac{1}{1+r} = 0 \tag{2.15}
$$

We solve (2.14) for $\lambda = c_1^{-\gamma}$ and plug this into (2.15) to obtain the Euler equation for the general CRRA case,

$$c_1^{-\gamma} = \beta(1+r)c_2^{-\gamma} \ . \tag{2.16}$$

You can easily verify that when setting $\gamma = 1$ in (2.16), we obtain the Euler equation for the special case of log preferences we found in (2.5).

We can use (2.16) to say something about consumption growth c_2/c_1 between periods 1 and 2 as a function of parameters. To this end, rewrite the Euler equation (2.16) as

$$\frac{c_2}{c_1} = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} \tag{2.17}$$

Since $\gamma > 0$, the above equation tells us that $c_2 > c_1$ whenever $\beta(1+r) > 1$, and conversely, $c_1 < c_2$ if $\beta(1+r) < 1$.

There is a more elegant expression that relates the *growth rate* of consumption to parameters. To derive it, recall the following approximation: if c_1 and c_2 are close, we have

$$\log(c_2/c_1) = \log\left(1 + \frac{c_2 - c_1}{c_1}\right) \approx \frac{c_2 - c_1}{c_1}$$

where the right-most term is the consumption growth rate. This follows because for x close to zero, it holds that $\log(1+x) \approx x$. Taking logs on both sides of (2.17), we have

$$\begin{aligned}
\frac{c_2 - c_1}{c_1} \approx \log(c_2/c_1) &= \log\left(\left[\beta(1+r)\right]^{\frac{1}{\gamma}}\right) \\
&= \frac{1}{\gamma}\Big[\log(1+r) + \log(\beta)\Big] \\
&\approx \frac{1}{\gamma}\left[r + \log(\beta)\right]
\end{aligned} \tag{2.18}$$

The last step follows since for small r we have $\log(1+r) \approx r$. We can go one step further if we define $\beta \equiv \frac{1}{1+\rho}$, where ρ is the *subjective discount rate*, also called the *rate of time preference*. Then

$$\log(\beta) = \log\left(\frac{1}{1+\rho}\right) = -\log(1+\rho) \approx -\rho$$

and we can write (2.18) as

$$\frac{c_2 - c_1}{c_1} \approx \frac{1}{\gamma}(r - \rho) \ . \tag{2.19}$$

The relation (2.19) is straightforward to interpret:

- If $r > \rho$, the market interest rate is higher than the households time preference rate, and the households shifts consumption to period 2. Consumption growth must therefore be positive.

- If $r = \rho$, the market discounts the future at exactly the same rate as the household, so consumption growth is zero. We have that $c_2 = c_1$.

- If $r < \rho$, the household discounts the future more heavily than the market and will want to front-load consumption, hence the consumption growth rate is negative.

How strongly the household responds to the difference $r - \rho$ is governed by $\frac{1}{\gamma}$, which we discuss next. ■

We are now in a position to quantify the willingness of a consumer to substitute between consumption in periods 1 and 2. This willingness is captured by the *elasticity of intertemporal substitution* which determines how relative consumption c_2/c_1 depends on the interest rate. As economists, we prefer such a measure to be expressed as an elasticity, i.e., we want to find the percent change in c_2/c_1 when $(1+r)$ changes by one percent.

Recall that an elasticity is a unit-free measure which tells us by how many percent some variable y changes if x changes by one percent, i.e.,

$$\underbrace{dy/y}_{\text{\% change in } y} = Elasticity \times \underbrace{dx/x}_{\text{\% change in } x}$$

We can rewrite this expression in various equivalent ways to compute the elasticity:

$$Elasticity = \frac{dy/y}{dx/x} = \frac{dy}{dx}\frac{x}{y} = \frac{d\log y}{d\log x}$$

The last equality follows if you recall that $d\log y = \frac{1}{y}dy$ and is probably the most widely used method to compute elasticities in economics.

Returning to our problem, we want to quantify the elasticity of relative consumption with respect to interest rate changes, which we call the *elasticity of intertemporal substitution* (EIS),[3]

$$EIS = \frac{d\log\left(c_2/c_1\right)}{d\log(1+r)} \,. \tag{2.20}$$

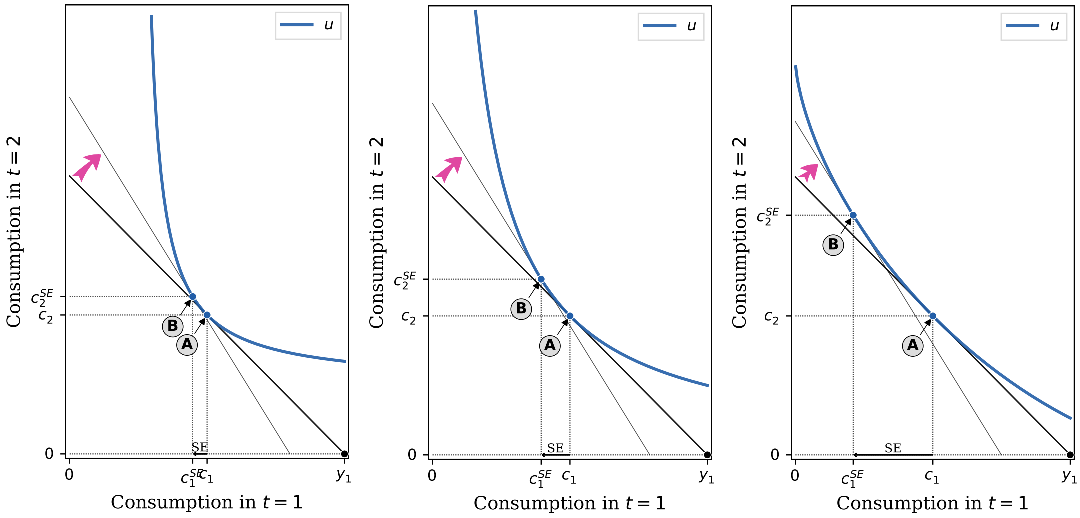
We need to evaluate this quantity at the *optimally chosen* consumption bundle (c_1, c_2) which satisfies the Euler equation (2.16). Taking logs on both sides of (2.16), we see that

$$\begin{aligned}
\log(c_2/c_1) &= \log\left(\left(\beta(1+r)\right)^{\frac{1}{\gamma}}\right) \\
&= \frac{1}{\gamma}\log(\beta(1+r)) \\
&= \frac{1}{\gamma}\log\beta + \frac{1}{\gamma}\log(1+r)
\end{aligned}$$

It is now straightforward to evaluate the elasticity in (2.20):

$$EIS = \frac{d\log\left(c_2/c_1\right)}{d\log(1+r)} = \frac{d\left[\frac{1}{\gamma}\log\beta + \frac{1}{\gamma}\log(1+r)\right]}{d\log(1+r)} = \frac{1}{\gamma} \tag{2.21}$$

[3] Note that some authors call it the *intertemporal elasticity of substitution* (IES).



(a) EIS = $\frac{1}{2}$ **(b)** EIS = 1 **(c)** EIS = 3

Figure 2.3: Substitution effect of an increase in r for different values of the elasticity of intertemporal substitution (EIS). Point **(A)** depicts the allocation at the initial interest rate. **(B)** shows the allocation at the new interest rate which yields the same utility level.

After all this work we can thus conclude that CRRA preferences are *isoelastic* — the elasticity of intertemporal substitution is always the same, and it is given by the inverse of the RRA parameter γ. What's more, you now see why the EIS is always one for log preferences where $\gamma = 1$.

Let's illustrate this finding using three parametrisations for γ shown in Figure 2.3:

(a) When the EIS is *low* (or equivalently, γ is high), the substitution effect is not very large, as illustrated in panel (a). The indifference curves have high curvature, so a tilt in the slope of the budget line is accommodated by only a small movement from point **(A)** to **(B)**.

(b) With *log preferences*, the EIS is always one. This case is depicted in panel (b).

(c) When the EIS is *high* (or equivalently, γ is low), the substitution effect is large. The indifference curves have low curvature, so even a small change in r can move the consumption allocation far from its initial level. This case is shown in panel (c).

There is another useful takeaway from the above exercise: the EIS also approximately measures changes in *consumption growth* $(c_2 - c_1)/c_1$ as r changes. This follows immediately from the approximation we derived in (2.19). Taking the derivative with respect to r, we see that

$$\frac{d\big((c_2 - c_1)/c_1\big)}{dr} \approx \frac{1}{\gamma} \tag{2.22}$$

This tells us that if the interest rate increases by one percentage point, consumption growth between periods 1 and 2 increases by approximately $\frac{1}{\gamma}$ percentage points!

Example 2.3 (Consumption responses to change in r for different EIS). Consider the setting in Example 2.2 with CRRA preferences, and assume that the household receives income $y_1 = 3$ and $y_2 = 1$. The discount factor is set to $\beta = 1$ and the interest rate in the baseline scenario is $r = 0$.

We know that with this parametrisation, the household will perfectly smooth consumption and choose $c_1 = c_2 = 2$ irrespective of the value of γ.

How does optimal consumption respond when the interest rate increases to $r' = 0.01$? We consider three different scenarios with $\gamma = 2, 1, \frac{1}{2}$, which correspond to an EIS of $\frac{1}{\gamma} = \frac{1}{2}, 1, 2$, and solve the household problem for each parametrisation and interest rate.

Given that the gross interest rate $(1 + r)$ increases by 1%, we know from (2.21) that the ratio c_2/c_1 should increase by about 0.5%, 1% and 2% for these three EIS values. Moreover, from (2.22) we know that consumption growth from period 1 to 2 should increase by 0.5, 1 and 2 percentage points, respectively. The exact figures are reported in Table 2.2, which are very close to what we'd expect given the EIS.

	r	c_1	c_2	$\frac{c_2}{c_1}$	$\%\Delta\frac{c_2}{c_1}$	$\frac{c_2-c_1}{c1}$	$\Delta\frac{c_2-c_1}{c1} \times 100$
EIS = 0.5	0.0%	2.000	2.000	1.000	–	0.00%	–
	1.0%	2.000	2.010	1.005	0.50%	0.50%	0.50
EIS = 1.0	0.0%	2.000	2.000	1.000	–	0.00%	–
	1.0%	1.995	2.015	1.010	1.00%	1.00%	1.00
EIS = 2.0	0.0%	2.000	2.000	1.000	–	0.00%	–
	1.0%	1.985	2.025	1.020	2.01%	2.01%	2.01

Table 2.2: Consumption responses to a 1% increase in the interest rate for different EIS values. Columns c_1 and c_2 report the optimal consumption allocation for each interest rate and EIS. Column $\%\Delta\frac{c_2}{c_1}$ shows the relative change in $\frac{c_2}{c_1}$ as the interest rate changes. Column $\frac{c_2-c_1}{c_1}$ lists the consumption growth between periods 1 and 2. The last column shows how consumption growth responds to the change in r (in percentage points).

■

2.3 Life cycle model with many periods

So far, we have only dealt with two-period models. If we assume that the household gets income only in the first period, this can be interpreted as a very stylised model of a life cycle in which the household receives labour income during working age, then retires and lives off its savings. Figure 2.4 depicts this situation, assuming that the household perfectly smooths consumption across both periods. In period 1, the household accumulates savings which are used to finance consumption in period 2.

In the previous example, we assumed that the agent lives until age $T = 2$. Of course, we can extend this setting to something more realistic such as $T = 60$, where the agent lives for 60 years (most economic models start at an age of around 20 since we are usually interested to understand economic decisions of financially independent adults).

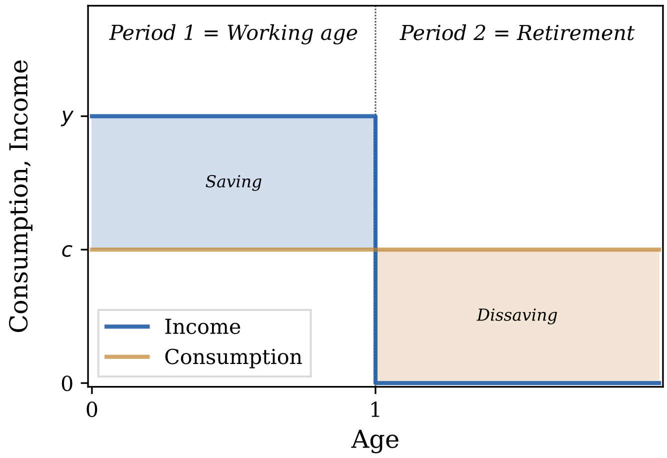


Figure 2.4: Stylised two-period life cycle model. The household receives income only in the first period, whereas consumption is perfectly smoothed across both periods (assuming $r = 0$ and $\beta = 1$).

A stylised version of such a model which goes back to Modigliani and Brumberg (1954) is shown in Figure 2.5, where we continue to assume that $\beta = 1$, $r = 0$ and additionally impose a constant labour income throughout working age. Note that because of $r = 0$,

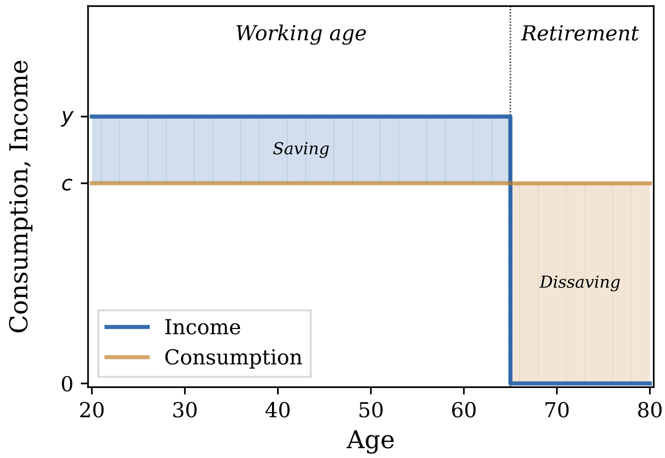


Figure 2.5: Stylised 60-period life cycle model. Income is constant during working age and zero in retirement.

the areas labelled "saving" and "dissaving" must be of equal size to satisfy the lifetime budget constraint.

We formalise this model following the exposition in Jappelli and Pistaferri (2017, chapter 1). Assume the household lives for T periods and receives income y_t at age $t = 0, \ldots, T-1$.[4] For each t, it chooses consumption c_t and the amount of assets a_{t+1} to

[4] In multi-period settings the convention is to denote the initial point in time as $t = 0$. You can think of t as

bring to the next period. For some initial asset level a_0, the maximisation problem reads as follows:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t) \tag{2.23}$$

$$\text{s.t.} \quad c_t + a_{t+1} = (1+r)a_t + y_t \qquad \forall\ t \tag{2.24}$$

$$a_T \geq 0, \quad a_0 \text{ given} \tag{2.25}$$

We additionally impose that the household cannot leave behind debt when it exists the economy, hence $a_T \geq 0$.

We first want to consolidate the period-specific budget constraints (2.24) into a single present-value lifetime budget constraint as we did for the two-period case. Intuitively, we already know that it will state that lifetime consumption cannot exceed lifetime wealth (initial assets + income). The formal derivation requires a few steps which we relegate to appendix 2.1 since it's a lot of algebra but adds little to our understanding of the problem. We instead directly state the lifetime budget constraint which is given by

$$\underbrace{\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t}}_{\text{PV of cons.}} = \underbrace{(1+r)a_0}_{\text{Init. wealth}} + \underbrace{\sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t}}_{\text{PV of income}} \tag{2.26}$$

The effective initial wealth is $(1+r)a_0$ instead of just a_0 which looks unusual but does not change the problem at hand.[5]

We are now ready to set up the Lagrangian, which is given by

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[(1+r)a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} - \sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} \right] \tag{2.27}$$

The first-order condition with respect to any c_t is

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \frac{\lambda}{(1+r)^t} = 0$$

To obtain the Euler equation, we need to eliminate λ using another equation, for example the FOC for c_{t+1}:

$$\frac{\partial \mathcal{L}}{\partial c_{t+1}} = \beta^{t+1} u'(c_{t+1}) - \frac{\lambda}{(1+r)^{t+1}} = 0$$

Solving both of these for λ and setting them equal, we get

$$\beta^t (1+r)^t u'(c_t) = \beta^{t+1} (1+r)^{t+1} u'(c_{t+1})$$

age, where the household enters the economy at (adult) age zero and exits the economy at the moment it attains age T, so the last period of life starts at age $T-1$.

[5] The budget constraint (2.24) for $t = 0$ reads $c_0 + a_1 = (1+r)a_0 + y_0$, i.e., the household instantly earns interest income on its initial assets. While this looks strange, it allows us to use the same notation as for the two-period model.

Dividing by $\beta^t(1+r)^t$ yields the final version of the Euler equation:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

This is the usual intertemporal optimality condition linking two adjacent periods t and $t+1$ which generalises naturally from the two-period setting. Assuming that preferences are CRRA, we get the familiar expression

$$c_t^{-\gamma} = \beta(1+r)c_{t+1}^{-\gamma} . \tag{2.28}$$

Solving the remainder of the problem at this level of generality involves significant amounts of algebra, so we again leave this to the appendix. Instead, we'll look at the special case of log preferences, constant labour income and retirement.

Example 2.4 (Simple model with retirement). Let's consider the simplest multi-period model of retirement which generates the income and consumption profiles illustrated in Figure 2.5. Assume that the household lives for T periods, works for the first N periods, receives a constant income y while working and starts with zero initial assets, $a_0 = 0$. We set $\beta = 1$ and $r = 0$ as this considerably simplifies the solution.

With this parametrisation, the Euler equation (2.28) reads

$$c_t^{-\gamma} = \beta(1+r)c_{t+1}^{-\gamma} \implies c_t^{-\gamma} = c_{t+1}^{-\gamma} \implies c_t = c_{t+1}$$

for all t, thus consumption is identical in each period which we denote as $c_t = c$. Moreover, since there is no discounting, lifetime consumption is given by

$$\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{T-1} c = Tc . \tag{2.29}$$

Lastly, lifetime wealth becomes

$$(1+r)a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} = \sum_{t=0}^{N-1} y = Ny . \tag{2.30}$$

where we use the fact that income is zero for all $t \geq N$. We can plug (2.29) and (2.30) into the lifetime budget constraint (2.26) to solve for consumption:

$$T \cdot c = N \cdot y \implies c = \frac{N}{T}y \tag{2.31}$$

The household thus consumes a constant fraction $\frac{N}{T} < 1$ of income while working and saves the remaining fraction $(1 - \frac{N}{T})$. It therefore accumulates assets over the life cycle which are then used to finance consumption in retirement. This asset trajectory is shown in Figure 2.6: intuitively, it peaks exactly in the period when the household enters retirement, and approaches zero as the household comes closer to the terminal period $T-1$.

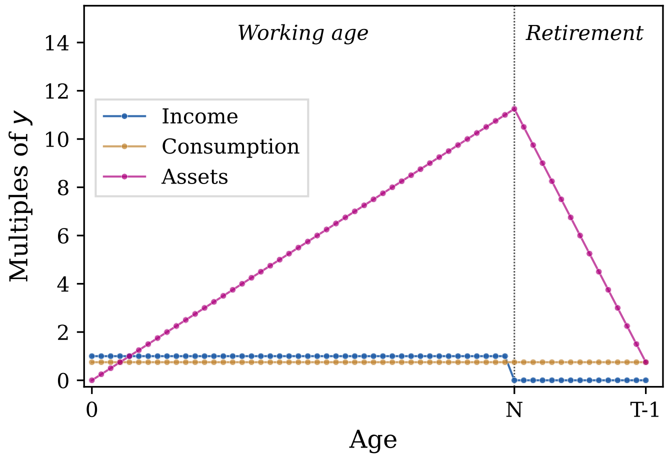


Figure 2.6: Life cycle profiles of income, consumption and assets for model with log preferences, $r = 0$ and $\beta = 1$. Dots indicate choices at each age.

■

While the previous example was not too difficult to solve, the complexity of these solutions increases considerably if we abandon more and more of the simplifying assumptions. One extension that can still be solved with reasonable amounts of work is discussed next.

Example 2.5 (Life cycle with discounting). Continuing with the setting in Example 2.4, we now impose log preferences and discounting with $\beta < 1$, but keep the interest rate at $r = 0$. The remaining parameters are unchanged.

The Euler equation linking consumption in two consecutive periods now becomes

$$c_t^{-1} = \beta c_{t+1}^{-1} .$$

Rewriting this as $c_{t+1} = \beta c_t$, we can express consumption in t as a function of c_0 by repeated substitution:

$$c_t = \beta^t c_0 . \tag{2.32}$$

Substituting this into the right-hand side of the LTBC (2.26), we see that

$$\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{T-1} \beta^t c_0 = c_0 \sum_{t=0}^{T-1} \beta^t = c_0 \frac{1-\beta^T}{1-\beta}$$

where the last step follows from the summation formula for the first T terms of a geometric series (recall that $\beta \neq 1$ which is a condition to apply this rule). The left-hand side of the LTBC remains unchanged and is equal to (2.30). The LTBC thus reads

$$c_0 \frac{1-\beta^T}{1-\beta} = Ny .$$

We can solve for c_0 to find

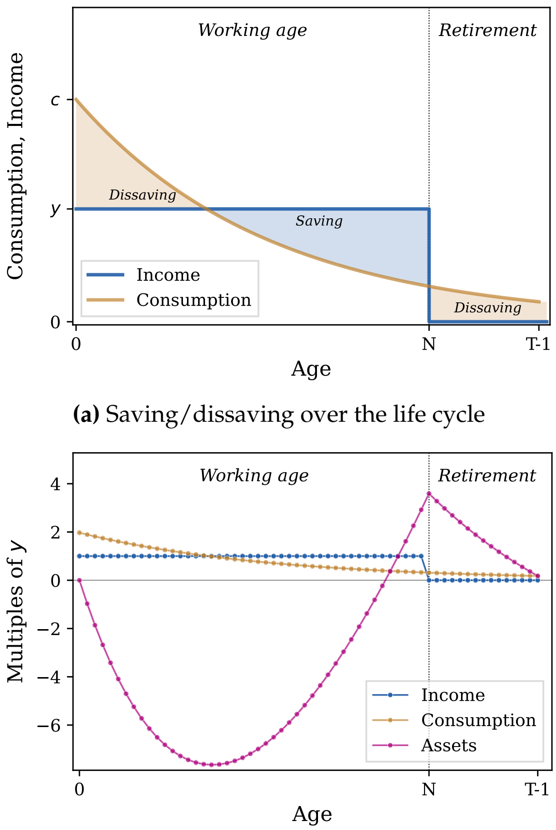
$$c_0 = \frac{1-\beta}{1-\beta^T} Ny = \frac{1}{1+\beta+\beta^2+\cdots+\beta^{T-1}} Ny \tag{2.33}$$

For $\beta = 1$, the expression on the right-hand side is equal to $\frac{N}{T}y$ and we are back to the case we found in Example 2.4. For $\beta < 1$, we have that

$$c_0 = \frac{1}{1+\beta+\beta^2+\cdots+\beta^{T-1}} Ny > \frac{1}{T} Ny$$

so the more impatient household brings forward consumption to the first period. Consumption in later periods can be obtained from (2.32).

Figure 2.7 plots the solution to this problem if we assume that $T = 60$, $N = 45$, $y = 1$ and $\beta = 0.96$. This discount factor is a standard value for macroeconomic models at annual frequency. As you can see in panel (a), because the household is impatient relative to the interest rate, it borrows in the first ≈ 15 periods of its life and starts saving for retirement thereafter. Panel (b) shows the resulting asset profile over the life cycle.



(a) Saving/dissaving over the life cycle

(b) Profiles for income, consumption and assets

Figure 2.7: Life cycle profiles of income, consumption and assets for model from Example 2.5 with log preferences, $r = 0$ and $\beta = 0.96$. Dots indicate choices at each age.

■

It is usually instructive to check that a general solution like the one in Example 2.5 simplifies to a nested, more restrictive case. This also helps verify that our solution is correct. For example, recall the problem from (2.1) which is identical to Example 2.5 if we set $T = 2$ and $N = 1$. Inspecting the right-most expression in (2.33), we see that for $T = 2$ and $N = 1$ it simplifies to

$$c_0 = \frac{1}{1+\beta} y$$

which is exactly the first-period consumption we found in (2.6).

At this point, it is not worthwhile to generalise the above framework further. While it is possible to solve for CRRA preferences with $\gamma \neq 1$ (see appendix), the resulting analytical solution does not yield many additional insights. Most other extensions used in macroeconomics and household finance have no closed-form solution at all but are instead solved numerically on the computer. For the remainder of this section, we therefore interpret the output generated from numerical solutions.

Example 2.6 (Consumption growth and EIS). Consider the life cycle model in (2.23) with $a_0 = 0$, $T = 60$ and a working life of $N = 45$ periods with constant income $y_t = 1$ for all $t < N$. We set the discount factor to $\beta = 0.96$ and the interest rate to $r = 0.05$. Because $(1 + r) > \beta^{-1}$, we know that the household has the incentive to save initially and shift consumption to later periods. To see this, solve the Euler equation (2.28) for consumption growth,

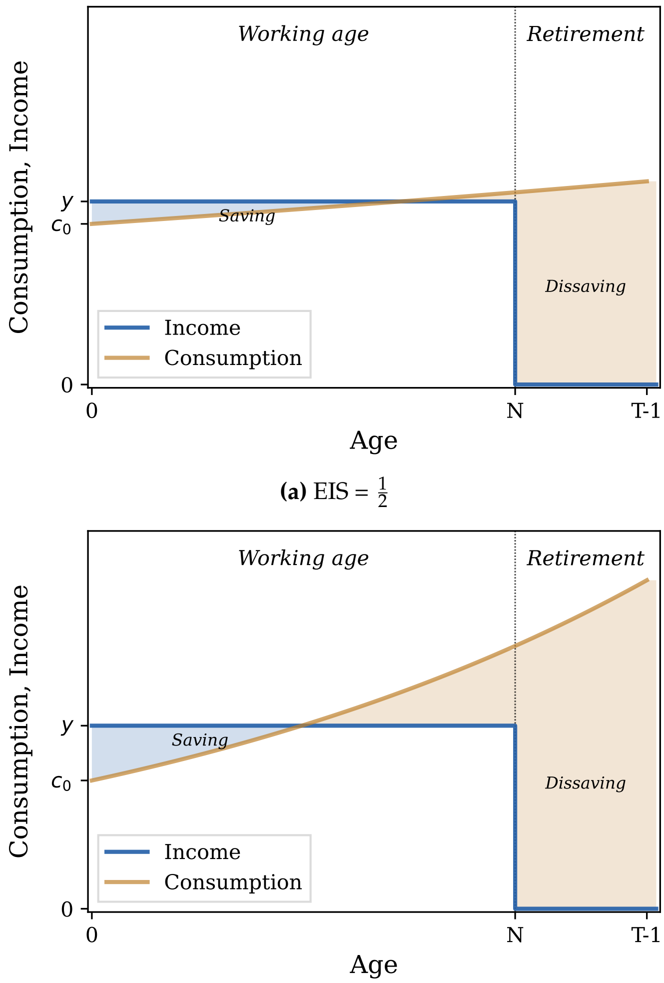
$$\frac{c_{t+1}}{c_t} = \Big[\beta(1+r)\Big]^{\frac{1}{\gamma}} > 1$$

Since $\gamma > 0$ and $\beta(1 + r) > 1$, the gross consumption growth c_{t+1}/c_t is larger than one. However, the exact slope of the consumption profile depends on the EIS, $\frac{1}{\gamma}$.

We consider two cases, EIS = $\frac{1}{2}$ and EIS = 2. In the first case, consumption growth will be moderate, whereas in the second case it will be more pronounced. These scenarios are shown in Figure 2.8. For consumption to grow faster over the life cycle, the household first needs to consume little and accumulate assets which are used to finance higher consumption at the later stages of life. The asset profiles in Figure 2.9 illustrate this situation. For the highly elastic case, the household accumulates substantially more assets over its working life. ■

So far, we imposed that income is constant during working life. In reality, most people face an upward-sloping income profile as they gather work experience and receive promotions. We conclude this section with a more realistic model which takes this into account.

Example 2.7 (Life cycle with earnings growth). Consider the life cycle model in (2.23) with $a_0 = 0$, $T = 60$ and a working life of $N = 45$ periods. Initially, the consumer faces an upward-sloping income profile illustrated in Figure 2.10. This profile is taken from Cocco, Gomes, and Maenhout (2005), a seminal paper in the household finance literature which estimates income trajectories from the PSID for US men with high school



(a) EIS = $\frac{1}{2}$

(b) EIS = 2

Figure 2.8: Income and consumption profiles for different EIS values in Example 2.6 with $\beta = 0.96$ and $r = 0.05$.

education. After retirement, the agent receives retirement benefits which correspond to 68% of pre-retirement earnings. The earnings profile is rescaled so that it is on average one during working life.

We assume that $r = 0.04$ and $\beta = \frac{1}{1+r}$ so that the household prefers to perfectly smooth consumption over the life cycle. To do so, it initially borrows against future earnings and starts saving only in its mid-thirties, as shown in Figure 2.10. ■

The prediction from this more realistic calibration is that we should see households in debt until their late forties, and their wealth should peak at retirement. They then completely decumulate their wealth holdings during the retirement phase. Is this what we observe in the data?

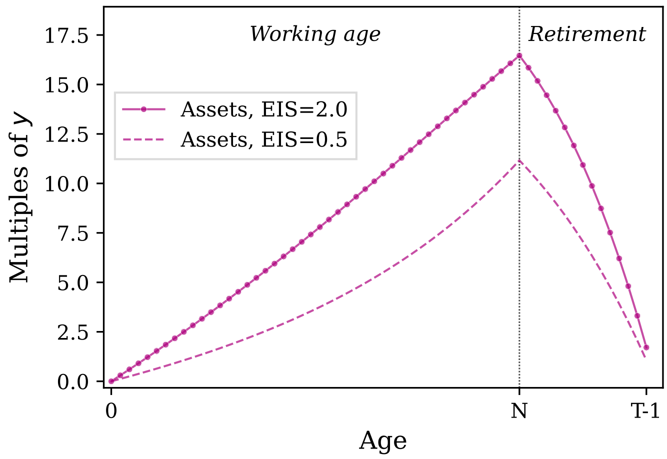


Figure 2.9: Life cycle profiles for assets for different EIS values in Example 2.6 with $\beta = 0.96$ and $r = 0.05$.

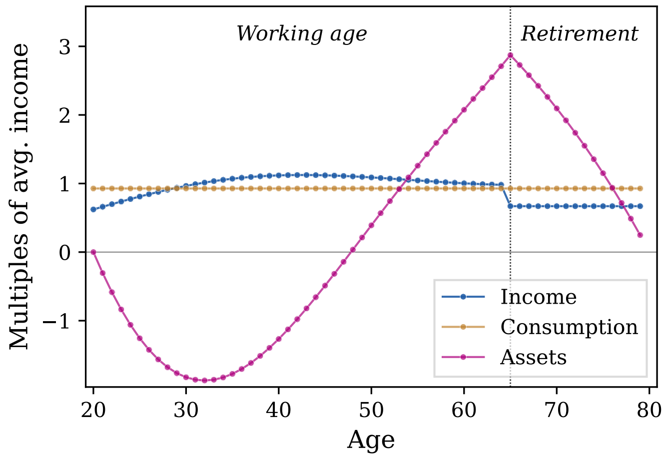


Figure 2.10: Life cycle profiles for income, consumption and assets for Example 2.7.

2.4 Life cycle profiles in the data

In the previous section, we saw that unless there was a gap between the subjective rate of time preference and the market interest rate, consumption would end up being constant over the life cycle. This was even true when income was not constant or when there was no income in retirement. Households could perfectly smooth consumption because we did not impose any borrowing constraints, and their net asset position evolved in whichever way needed to generate this flat consumption profile (subject to the lifetime budget constraint). At any particular age, income and consumption were therefore more or less disconnected.

Is there any evidence that this prediction holds up in the data? Figure 2.11 plots average household income and nondurable consumption by education group for the

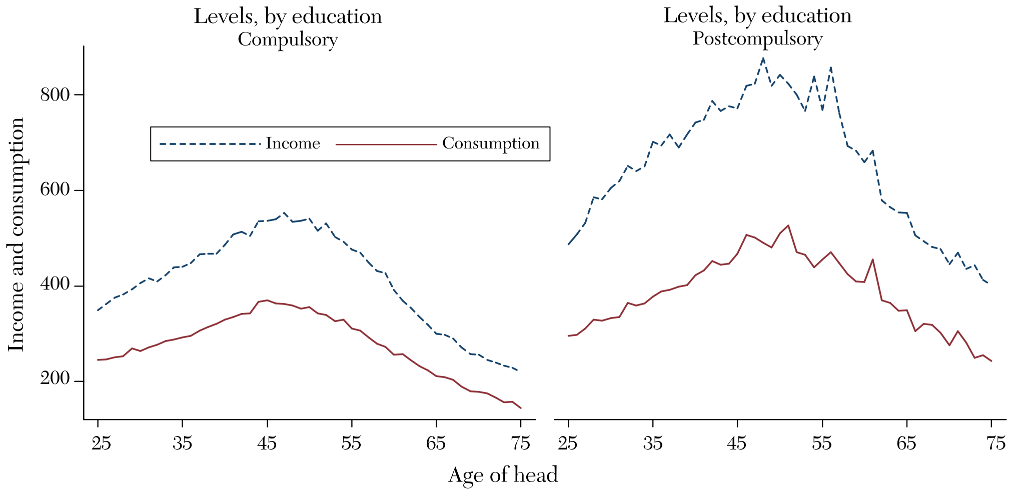


Figure 2.11: Average income and (nondurable) consumption by education in £/week. Source: Attanasio and Weber (2010, Figure 1), based on UK Family Expenditure Survey 1978–2007

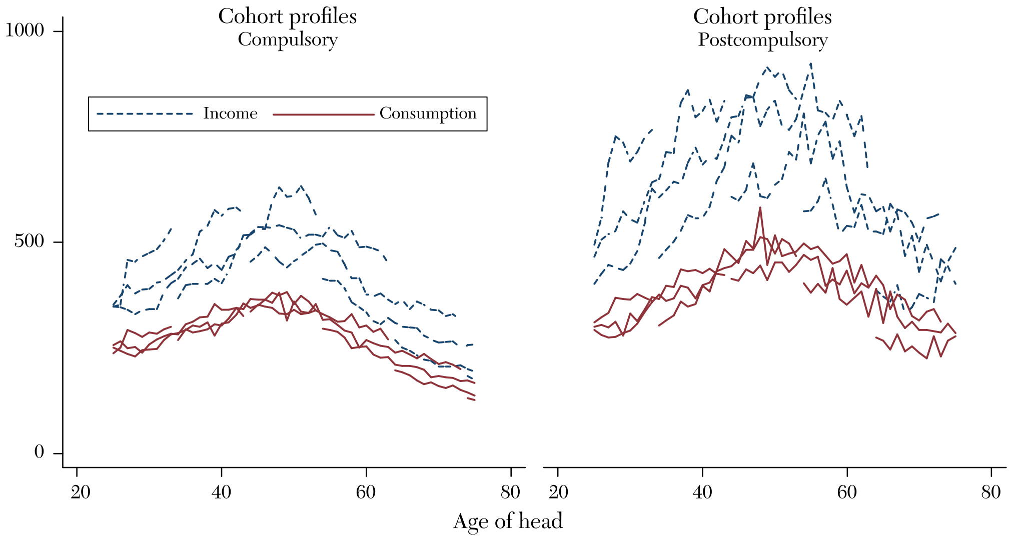


Figure 2.12: Average income and (nondurable) consumption *by cohort* and education in £/week. Source: Attanasio and Weber (2010, Figure 2), based on UK Family Expenditure Survey 1978–2007

UK. From this graph it is clear that consumption tracks the concave shape of income over the life cycle, suggesting that households cannot perfectly smooth consumption.

However, as Attanasio and Weber (2010) point out, this graph lumps together different cohorts at the same age which is problematic since younger cohorts are richer due to economic growth. Figure 2.12 additionally disaggregates the data by birth cohort, and the cohort-specific consumption series are indeed somewhat flatter. Finally, Figure 2.13 shows *per capita* values, since household consumption changes over the life cycle when

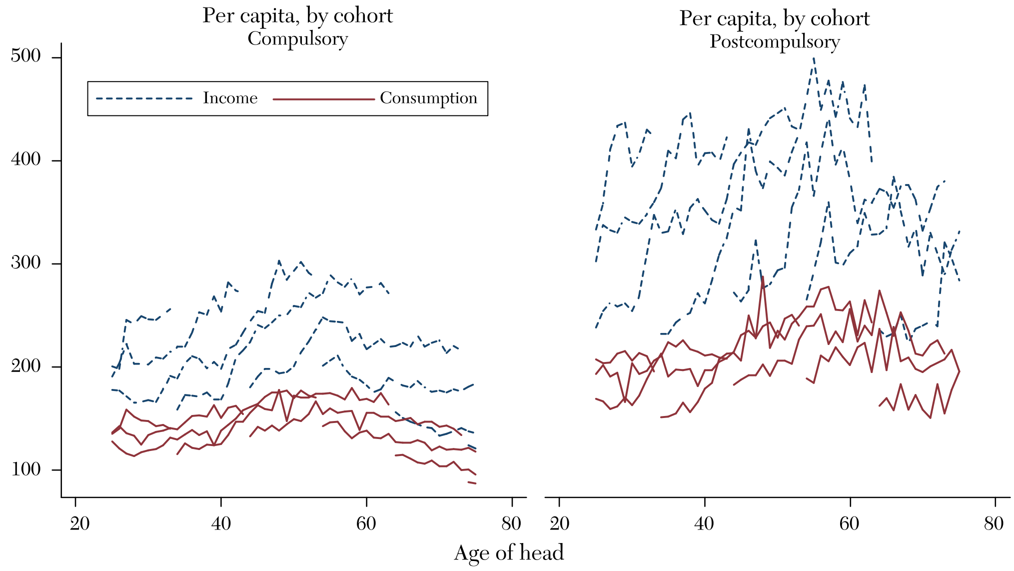


Figure 2.13: Average *per capita* income and (nondurable) consumption by cohort and education in £/week. Source: Attanasio and Weber (2010, Figure 3), based on UK Family Expenditure Survey 1978–2007

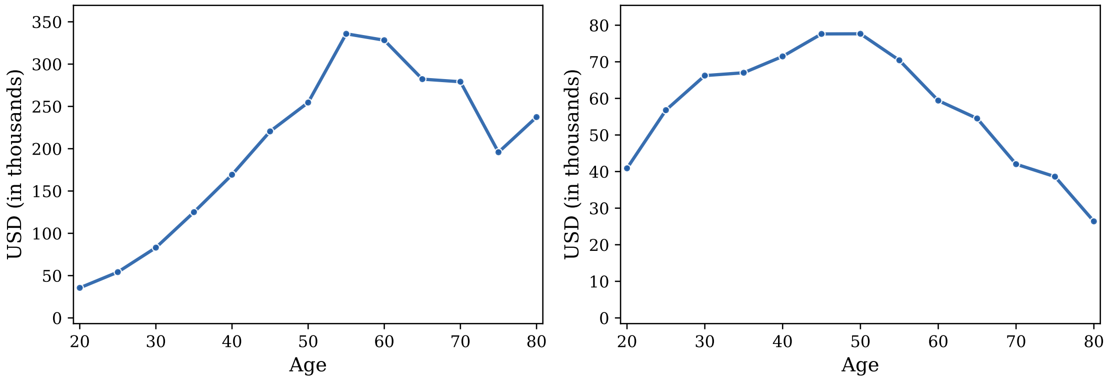
children are born into the household or move out. Controlling for household size eliminates part of the consumption hump observed observed around the age of 50. After applying these corrections, there seems to be some evidence for consumption smoothing, even though not to the extent suggested by our models.

Nonetheless, the asset profiles observed in the data look quite different from what we have found in Figure 2.10 and throughout this unit. Figure 2.14, panel (a) plots the median household net worth (including housing) from the Survey of Consumer Finances for the US, which turns out to be positive at all ages. This asset profile does not look anything like the one in Figure 2.10, even though in the data households also face an initially upward-sloping earnings profile as shown in Figure 2.14, panel (b), just like we assumed in Example 2.7. Our theory predicts that households should borrow against future income, suggesting that in the real world there may be credit market frictions preventing them to do so.

Another discrepancy arises if we compare the decumulation of wealth predicted by our model (e.g., in Figure 2.10) compared to the data. Figure 2.14 shows that the median household keeps a lot of wealth until very old age instead of consuming it. This might be due to several reasons such as the intention to leave bequests for one's children, or because a substantial fraction of that wealth is invested in a household's primary residence, an illiquid asset people cannot or do not want to downsize.

2.5 Main takeaways

In terms of theory, this unit covered the following concepts:



(a) Net worth

(b) Labour income

Figure 2.14: Median net worth and gross household labour income (incl. retirement benefits) in thousands of 2009 USD. Medians are computed within 5-year age bins. Data source: SCF 1998–2007

1. We studied how changes in the interest rate affect consumption in a two-period model and how such changes can be decomposed into three distinct effects:
    - The *substitution effect* quantifies how consumption shifts between periods due to changes in the relative price.
    - The *income effect* characterizes changes to an agent's budget arising from higher interest income (lender) or higher interest payments (borrower).
    - Lastly, the *wealth effect* quantifies how the present value of future income is affected by the interest rate.
2. We introduced the *elasticity of intertemporal substitution* as a way to quantify how strongly households adjust their consumption growth in response to interest rate changes.
3. We discussed the *life cycle model* with many periods as a natural extension of the two-period problem. In this model, households save during their working life to finance consumption in retirement.

Moreover, we examined whether the life cycle model can explain the trajectories of consumption, income and wealth in the data.

1. We found that there was some support for consumption smoothing in the data, even though to a lesser extent than what is predicted by our (simple) model.
2. Our simple model performed poorly in terms of explaining the life cycle profile of wealth. In particular, it failed to capture dissaving in old age.

Appendix 2.1: Full derivation of the life cycle model

This section contains the detailed steps required to solve the full life cycle model with many periods and CRRA preferences. We start with the general problem stated in (2.23), derive the lifetime budget constraint and then solve for optimal consumption.

Lifetime budget constraint. Take the budget constraint (2.24) for any t and rewrite it as

$$a_{t+1} = (1+r)a_t + y_t - c_t \,.$$

Dividing this by $(1+r)^{t+1}$, we have

$$\begin{aligned}\frac{a_{t+1}}{(1+r)^{t+1}} &= \frac{(1+r)a_t}{(1+r)^{t+1}} + \frac{y_t}{(1+r)^{t+1}} - \frac{c_t}{(1+r)^{t+1}} \\ &= \frac{a_t}{(1+r)^t} + \frac{1}{1+r}\frac{y_t}{(1+r)^t} - \frac{1}{1+r}\frac{c_t}{(1+r)^t}\end{aligned}$$

Next, we sum both sides over all $t = 0, \dots, T-1$,

$$\sum_{t=0}^{T-1} \frac{a_{t+1}}{(1+r)^{t+1}} = \sum_{t=0}^{T-1} \frac{a_t}{(1+r)^t} + \frac{1}{1+r}\sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} - \frac{1}{1+r}\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} \tag{2.34}$$

The sum on the left-hand side can be written as

$$\sum_{t=0}^{T-1} \frac{a_{t+1}}{(1+r)^{t+1}} = \sum_{t=1}^{T} \frac{a_t}{(1+r)^t} = \sum_{t=1}^{T-1} \frac{a_t}{(1+r)^t}$$

where the last step follows because $a_T = 0$ by (2.25) and optimality (the household would not want to leave behind a positive amount of assets). We can subtract the right-hand side of (2.34) from the first term on left-hand side to see that

$$\sum_{t=0}^{T-1} \frac{a_t}{(1+r)^t} - \sum_{t=0}^{T-1} \frac{a_{t+1}}{(1+r)^{t+1}} = \sum_{t=0}^{T-1} \frac{a_t}{(1+r)^t} - \sum_{t=1}^{T-1} \frac{a_t}{(1+r)^t} = a_0$$

The LTBC in (2.34) therefore becomes

$$0 = a_0 + \frac{1}{1+r}\sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} - \frac{1}{1+r}\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t}$$

Multiplying by $(1+r)$ and rearranging terms, we obtain the final form of the present-value lifetime budget constraint:

$$\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} = (1+r)a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} \tag{2.35}$$

Optimal consumption. We start with the Euler equation (2.28) from the main text. If we write it out explicitly for $t = 0, 1, \ldots,$ we have

$$\begin{aligned}
c_0^{-\gamma} &= \beta(1+r)c_1^{-\gamma} && \Longrightarrow && c_1 = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} c_0 \\
c_1^{-\gamma} &= \beta(1+r)c_2^{-\gamma} && \Longrightarrow && c_2 = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} c_1 \\
&\vdots && && \vdots
\end{aligned}$$

We can thus use repeated substitution to link periods that are farther apart, for example:

$$c_2 = \left[\beta(1+r)\right]^{\frac{2}{\gamma}} c_0$$

More generally, we have

$$c_t = \left[\beta(1+r)\right]^{\frac{t}{\gamma}} c_0 \tag{2.36}$$

Substituting for the c_t inside the sum in (2.35), we therefore get

$$\begin{aligned}
\sum_{t=0}^{T-1} \frac{c_t}{(1+r)^t} &= \sum_{t=0}^{T-1} \frac{\left[\beta(1+r)\right]^{\frac{t}{\gamma}} c_0}{(1+r)^t} \\
&= c_0 \sum_{t=0}^{T-1} \beta^{\frac{t}{\gamma}} (1+r)^{\frac{t}{\gamma}-t} \\
&= c_0 \sum_{t=0}^{T-1} \beta^{\frac{t}{\gamma}} (1+r)^{\frac{t(1-\gamma)}{\gamma}} \\
&= c_0 \sum_{t=0}^{T-1} \left[\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}\right]^t
\end{aligned}$$

When the model's parameters are such that

$$\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} \neq 1$$

we can apply the summation formula for the first T terms of a geometric series to find that

$$\sum_{t=0}^{T-1} \left[\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}\right]^t = \frac{1 - \left(\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}\right)^T}{1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}}$$

Summarising, from the Euler equation and the lifetime budget constraint we found that

$$c_0 \left[\frac{1 - \left(\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}\right)^T}{1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}}\right] = (1+r)a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t}$$

and therefore

$$c_0 = \underbrace{\left[\frac{1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}}{1 - \left(\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} \right)^T} \right]}_{\equiv \chi} \left[(1+r) a_0 + \sum_{t=0}^{T-1} \frac{y_t}{(1+r)^t} \right]$$

Consumption at age $t = 0$ is thus some fixed fraction χ of lifetime wealth which only depends on parameters and the interest rate. Consumption at any period $t > 0$ can then be determined from c_0 using (2.36).

Let's see whether this expression reduces to what we have previously found for various special cases. For example, for $T = 2$, $\gamma = 1$ and $a_0 = 0$, we are back to the two-period log-preferences case we studied in section 2.2. In this case, we see that

$$\left[\frac{1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}}{1 - \left(\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} \right)^T} \right] = \frac{1-\beta}{1-\beta^2} = \frac{1-\beta}{(1+\beta)(1-\beta)} = \frac{1}{1+\beta}$$

and therefore consumption in period 1 is given by

$$c_0 = \frac{1}{1+\beta} \left[y_0 + \frac{y_1}{1+r} \right] .$$

Reassuringly, this is exactly what we found in (2.12) after taking into account that time now starts at $t = 0$.

What about the three-period model with log preferences and $a_0 = 0$ which you are asked to solve in the exercises? In this case, the expression for χ simplifies to

$$\left[\frac{1 - \beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}}}{1 - \left(\beta^{\frac{1}{\gamma}} (1+r)^{\frac{1-\gamma}{\gamma}} \right)^T} \right] = \frac{1-\beta}{1-\beta^3} = \frac{1-\beta}{(1+\beta+\beta^2)(1-\beta)} = \frac{1}{1+\beta+\beta^2}$$

The polynomial factorisation in the denominator in the second step is not obvious but can be easily verified:

$$(1+\beta+\beta^2)(1-\beta) = 1+\beta+\beta^2-\beta-\beta^2-\beta^3 = 1-\beta^3$$

Consequently, consumption in the first period is given by

$$c_0 = \frac{1}{1+\beta+\beta^2} \left[y_0 + \frac{y_1}{(1+r)} + \frac{y_2}{(1+r)^2} \right] .$$

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Exercises

Exercise 2.1 (Income changes in a two-period model). Consider the standard two-period consumption-savings problem

$$\begin{aligned} \max_{c_1,\, c_2,\, a_2} \quad & \log(c_1) + \beta \log(c_2) \\ \text{s.t.} \quad & c_1 + a_2 = y_1 \\ & c_2 = (1+r)a_2 + y_2 \end{aligned}$$

with the following parameters: $\beta = 1$, $r = 0$, $y_1 = 1$ and $y_2 = 1.5$. Imagine now that the household gets a one-time bonus in the first period which increases income to $y_1 = 2$.

(a) Plot the initial and the new solution to the household's problem in a graph similar to Figure 2.2. How does the household's lending/borrowing behaviour change?

(b) Can you identify any substitution, income or wealth effects that explain the movement from the initial to the new allocation?

Exercise 2.2 (Substitution, income and wealth effects for a borrower). Recall the setting in Example 2.1: the illustration in Figure 2.2 depicted the substitution and income effects for a *lender* when the interest rate r increases.

(a) Assume now that the household is a *borrower* and draw the corresponding graphs for this scenario.

You don't need need to draw the graphs to scale, but if you want, you can assume the following parameters for this exercise: $\beta = 1$, $y_1 = 1$, $y_2 = 5$ and an interest rate change from $r = 0$ to $r' = 0.6$.

(b) Tabulate the sign of the income, substitution and wealth effects for a borrower in the same way as we did for the lender in Table 2.1. Can you unambiguously determine the direction of the total effect?

Exercise 2.3 (Three-period problem with log preferences). Assume a household lives for three periods. The first two periods represent its working age, so the household receives income (y_1, y_2), whereas the household is retired in the third period and receives no labour income. The maximisation problem reads as follows:

$$\begin{aligned} \max_{c_1, c_2, c_3, a_2, a_3} \quad & \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3) \\ \text{s.t.} \quad & c_1 + a_2 = y_1 && \text{(E.1)} \\ & c_2 + a_3 = (1+r)a_2 + y_2 && \text{(E.2)} \\ & c_3 = (1+r)a_3 && \text{(E.3)} \end{aligned}$$

Assume that the household can freely save or borrow between periods.

(a) Write down the present-value lifetime budget constraint (LTBC).

(b) Set up the Lagrangian for this problem, denoting the Lagrange multiplier for the LTBC by λ, and take the first-order derivatives w.r.t. c_1, c_2 and c_3.

(c) Use the first-order conditions to express c_2 and c_3 as functions of c_1 and parameters.

(d) Plug the expressions you found for c_2 and c_3 into the lifetime budget constraint and solve for the optimal choice of c_1.

(e) Let $\beta = 1$, $r = 0$, $y_1 = y_2 = 1$, and compute optimal period-1 consumption. What is the marginal propensity to consume (MPC) out of income in the first period? Compute and evaluate the derivative $MPC = \frac{\partial c_1}{\partial y_1}$, and provide an economic interpretation of the value you computed.

Exercise 2.4 (Life cycle with initial assets). Recall the simple T-period life cycle model from Example 2.4 where the household works the first N periods and retires thereafter. Specifically, the household maximises

$$\begin{aligned} \max_{\{c_t, a_{t+1}\}_{t=0}^{T-1}} \quad & \sum_{t=0}^{T-1} \log(c_t) \\ \text{s.t.} \quad c_0 + a_1 &= a_0 + y_0 \\ c_t + a_{t+1} &= (1+r)a_t + y_t \qquad \forall\, t > 0 \\ a_T \geq 0, \;\; & a_0 > 0 \;\text{ given} \end{aligned} \tag{E.4}$$

where income is constant during working life, i.e.,

$$y_t = \begin{cases} y & \text{if } t < N \\ 0 & \text{else} \end{cases}$$

Moreover, assume that the interest rate is $r = 0$. Note that we now have a budget constraint (E.4) that is specific to the first period — the only (cosmetic) difference is that the household no longer instantly receives interest income on its initial assets a_0.

(a) State the lifetime budget constraint for this problem.

(b) Derive the Euler equation.

(c) Use the Euler equation to conclude that consumption is constant across all periods, and denote this consumption level by c. Using the lifetime budget constraint, find an expression for c as a function of parameters.

(d) Compute the marginal propensity to consume (MPC) out of income y, i.e., derive the expression for $\partial c/\partial y$.

(e) Compute the marginal propensity to consume (MPC) out of initial wealth a_0, i.e., derive the expression for $\partial c / \partial a_0$. How does it compare to the MPC out of income y? Explain the economic intuition underlying this difference.

Exercise 2.5 (Life cycle with income growth). Consider a T-period life cycle model where the household works the first N periods and retires thereafter. The household maximises

$$
\begin{aligned}
\max_{\{c_t, a_{t+1}\}_{t=0}^{T-1}} \quad & \sum_{t=0}^{T-1} \log(c_t) \\
\text{s.t.} \quad & c_t + a_{t+1} = (1+r)a_t + y_t \qquad \forall\, t \\
& a_T \geq 0, \;\; a_0 = 0
\end{aligned}
$$

Income is assumed to grow by $z\%$ each period as long as the household is working, i.e.,

$$
y_{t+1} = (1+z)y_t \tag{E.5}
$$

and initial income is set to $y_0 = 1$. In retirement, $y_t = 0$ for all $t \geq N$. Moreover, assume that the interest rate is $r = 0$.

(a) Use the law of motion for income (E.5) and the initial condition to find an expression for y_t that depends only on y_0 and parameters.

(b) State the lifetime budget constraint for this problem. *Hint:* You will have to compute a sum of the form $\sum_{i=0}^{n} x^i$ for $x > 1$. The summation formula is given by[6]

$$
\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}
$$

(c) Derive the Euler equation.

(d) Use the Euler equation to conclude that consumption is constant across all periods, and denote this consumption level by c. Using the lifetime budget constraint, find an expression for c as a function of parameters.

(e) Let $T = 60$, $N = 45$ and $z = 0.02$. Compute the age at which the households starts saving for retirement, i.e., find the smallest t such that $y_t > c$!

[6] See https://en.wikipedia.org/wiki/Geometric_series#Closed-form_formula

3 Complete markets

3.1 Introduction

So far, our analysis of household behaviour was fully deterministic: households knew the realisation of their future income with certainty. To examine decisions under uncertainty, we need to assume that households cannot perfectly predict some future quantity, e.g., their income, the interest rates or even their survival chances in old age.

In this unit, we focus on uncertainty about a household's future income. We do this in a so-called *complete markets* environment in which households have access to a set of financial instruments which allow them to perfectly insure against any idiosyncratic risk. While this framework is unrealistic, it serves as a benchmark that can be compared to the incomplete markets model we will study in the next unit.

3.2 Uncertainty

Recall our canonical two-period consumption-savings problem (1.1) with deterministic (known) income in period two. When income is no longer certain, we say that it is *stochastic* or *random*. Formally, income is now modelled as a *random variable* with a known distribution, i.e., the household is uncertain about the exact realisation of future income, but it is perfectly aware of the underlying probability distribution and will consider the properties of this probability distribution when making optimal decisions.

In macroeconomics, we frequently model random variables using a well-known distribution such as the normal (Gaussian) or log-normal distribution. However, working with these distributions can quickly become complicated, so we will instead take an easier approach and assume that uncertain income takes on only two different realisations. We denote these realisations as "good" (with subscript g) or "bad" (with subscript b) with $y_b < y_g$:

$$y_{t+1} = \begin{cases} y_b & \text{with probability } \pi \\ y_g & \text{with probability } 1 - \pi \end{cases} \tag{3.1}$$

This definition says that y_{t+1} is a discrete random variable with two possible realisations: the bad outcome y_b has a probability of π, whereas with probability $1 - \pi$ the good outcome is observed, and π can be any real number between 0 and 1. As stated above, the household is perfectly aware of the values y_b, y_g and π. We say that the household has *rational expectations* because its beliefs coincide with the actual process governing y_{t+1}.

3.2.1 Mean and variance

We often characterize random variables using so-called *moments*. The first and second (central) moments are called the *mean* (or expected value) and the *variance*. For the random variable in (3.1), the mean can be computed as

$$\begin{aligned} \mathbb{E}_t y_{t+1} &= y_b \cdot \Pr\left(y_{t+1} = y_b \right) + y_g \cdot \Pr\left(y_{t+1} = y_g \right) \\ &= y_b \pi + y_g (1-\pi) \end{aligned} \tag{3.2}$$

The mean is thus the weighted sum of all possible outcomes with weights given by their corresponding probabilities (we use $\Pr\left(y_{t+1} = y_b \right)$ to denote the probability that y_b is realised). The expectations operator $\mathbb{E}_t y_{t+1}$ states that we are computing the expected value of its argument y_{t+1} using information available at time t. It tells us what value of y_{t+1} we can expect to see on average if we draw a large number of income realisations.

Conversely, the variance quantifies the dispersion of realisations around their mean. For random variable y_{t+1}, it is defined as[1]

$$\operatorname{Var}\left(y_{t+1} \right) = \mathbb{E}_t y_{t+1}^2 - \left(\mathbb{E}_t y_{t+1} \right)^2 \ . \tag{3.3}$$

Specifically, for y_{t+1} as defined in (3.1), we can evaluate the first term on the right, $\mathbb{E}_t y_{t+1}^2$, analogously to (3.2):

$$\mathbb{E}_t y_{t+1}^2 = y_b^2 \pi + y_g^2 (1-\pi)$$

while the second term is just the squared mean. Combining these expressions, we find that the variance of income is given by

$$\begin{aligned} \operatorname{Var}\left(y_{t+1} \right) &= \underbrace{y_b^2 \pi + y_g^2 (1-\pi)}_{\mathbb{E}_t y_{t+1}^2} - \underbrace{\left[y_b \pi + y_g (1-\pi) \right]^2}_{\left(\mathbb{E}_t y_{t+1} \right)^2} \\ &= y_b^2 \pi + y_g^2 (1-\pi) - y_b^2 \pi^2 - 2 y_b y_g \pi (1-\pi) - y_g^2 (1-\pi)^2 \\ &= y_b^2 \pi (1-\pi) + y_g^2 (1-\pi) \big(1 - (1-\pi) \big) - 2 y_b y_g \pi (1-\pi) \\ &= \pi (1-\pi) \left[y_b^2 + y_g^2 \right] - 2 y_b y_g \pi (1-\pi) \\ &= \pi (1-\pi) \left[y_b - 2 y_b y_g + y_g \right] \\ &= \pi (1-\pi) \left[y_b - y_g \right]^2 \end{aligned} \tag{3.4}$$

The expression in (3.4) has an intuitive interpretation: the variance increases with the squared distance between y_b and y_g. The farther these two outcomes are from each other, the more dispersed they are around their mean, and hence the variance increases.

Example 3.1. Consider the following stochastic income:

$$y_{t+1} = \begin{cases} y - \epsilon & \text{with probability } \frac{1}{2} \\ y + \epsilon & \text{with probability } \frac{1}{2} \end{cases} \tag{3.5}$$

[1] An alternative but equivalent definition of the variance is $\operatorname{Var}\left(y_{t+1} \right) = \mathbb{E}_t \left[\left(y_{t+1} - \mathbb{E}_t y_{t+1} \right)^2 \right]$.

for some $\epsilon > 0$. The bad and good outcomes are equally likely, each with a probability of $\pi = \frac{1}{2}$. From (3.2), it follows that the expected value is given by

$$\mathbb{E}_t\left[\, y_{t+1} \,\right] = \frac{1}{2}(y - \epsilon) + \frac{1}{2}(y + \epsilon) = y$$

while the variance can be obtained from (3.4) as

$$\mathrm{Var}\left(\, y_{t+1} \,\right) = \frac{1}{2}\left(1 - \frac{1}{2}\right)\left[y - \epsilon - (y + \epsilon)\right]^2 = \frac{1}{4}(2\epsilon)^2 = \epsilon^2$$

■

Example 3.2 (Mean-preserving spread). Continuing with Example 3.1, assume that stochastic income now follows

$$y_{t+1} = \begin{cases} y - 2\epsilon & \text{with probability } \frac{1}{2} \\ y + 2\epsilon & \text{with probability } \frac{1}{2} \end{cases} \tag{3.6}$$

It is straightforward to see that the mean remains unchanged, but the variance is instead given by

$$\mathrm{Var}\left(\, y_{t+1} \,\right) = 4\epsilon^2 \,.$$

Because the realisations are more spread out around the same mean, such a transformation is referred to as a *mean-preserving spread*. We will use it later to study how risk-averse agents change their behaviour if uncertainty increases while keeping the mean constant. ■

3.3 Risk aversion

Before solving a consumption-savings problem under uncertainty, we need to discuss what we mean by "risk aversion" and how this relates to our standard CRRA utility function,

$$u(c) = \begin{cases} \frac{c^{1-\gamma}-1}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases} \tag{3.7}$$

Generally speaking, the risk aversion is related to the utility function's curvature: recall Figure 1.1 which plots the CRRA utility function for various values of the relative risk aversion (RRA) parameter γ, reproduced here for convenience. As can be seen in Figure 3.1, more curvature is associated with higher values of γ.

In the next section, we characterise risk aversion using the certainty equivalent and the risk premium. An alternatively way to quantify risk aversion is to derive the Arrow-Pratt coefficient of relative risk aversion. It turns out that with CRRA preferences, the Arrow-Pratt coefficient of relative risk aversion is identical to the parameter γ in (3.7). This derivation is more advanced and therefore relegated to the appendix 3.1.

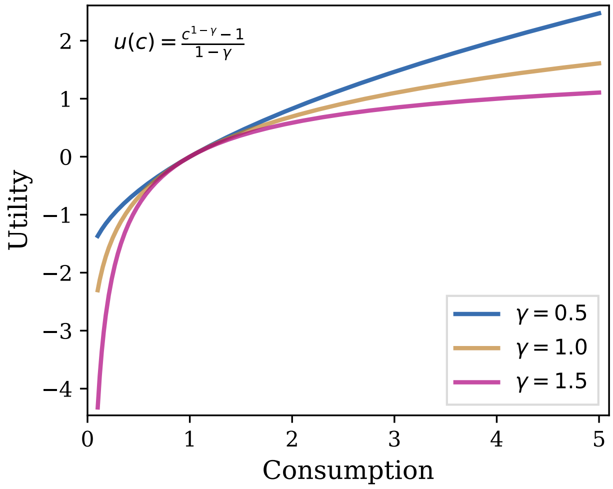


Figure 3.1: CRRA utility for different values of the relative risk aversion parameter γ.

3.3.1 Certainty equivalent and risk premium

One way to quantify risk aversion is to compute the certainty equivalent and the risk premium. Consider a static setting (no savings!) where the individual consumes c, which is stochastic. The individual can either have a bad draw c_b or a good draw c_g with $c_b < c_g$. For simplicity, let's impose that either of these outcomes is equally likely. The expected utility of such a gamble is thus given by

$$\mathbb{E}\left[u(c)\right] = \frac{1}{2}u(c_b) + \frac{1}{2}u(c_g) \tag{3.8}$$

where $u(\bullet)$ is the usual CRRA utility function. Imagine now that the individual could avoid the gamble and instead receive a deterministic amount CE. Which value of CE would yield the same utility as (3.8)? This value is called the *certainty equivalent* and is defined as

$$u(CE) = \mathbb{E}\left[u(c)\right] \tag{3.9}$$

Because the individual is risk averse, the amount CE is lower than the average consumption $\mathbb{E}\left[c\right]$ the individual can expect to receive: $CE < \mathbb{E}\left[c\right]$.[2]

The *risk premium* is defined as the difference between the expected outcome and the certainty equivalent:

$$p = \mathbb{E}\left[c\right] - CE$$

For a risk-averse consumer, the risk premium is positive.

The intuition is as follows: a risk-averse person would like to avoid risky gambles and therefore is willing to accept an amount that is certain but lower than the expected outcome of the gamble. The risk premium says that the consumer is willing to forfeit

[2]This follows from Jensen's inequality. Because the CRRA utility function $u(\bullet)$ is strictly concave, we have that $\mathbb{E}\left[u(c)\right] < u(\mathbb{E}\left[c\right])$ which implies $u(CE) < u(\mathbb{E}\left[c\right])$ from (3.9), and therefore $CE < \mathbb{E}\left[c\right]$.

an amount p in expectation. The more risk-averse an individual is, the higher the risk premium. This is illustrated in Figure 3.2, which graphically compares the certainty equivalents and risk premia for two individuals, one with a relative risk aversion $\gamma = 1$ (log preferences) and the other with $\gamma = 2$. Both face the same gamble, but as the figure shows, the more risk-averse person demands a higher risk premium (or accepts a lower certainty equivalent).

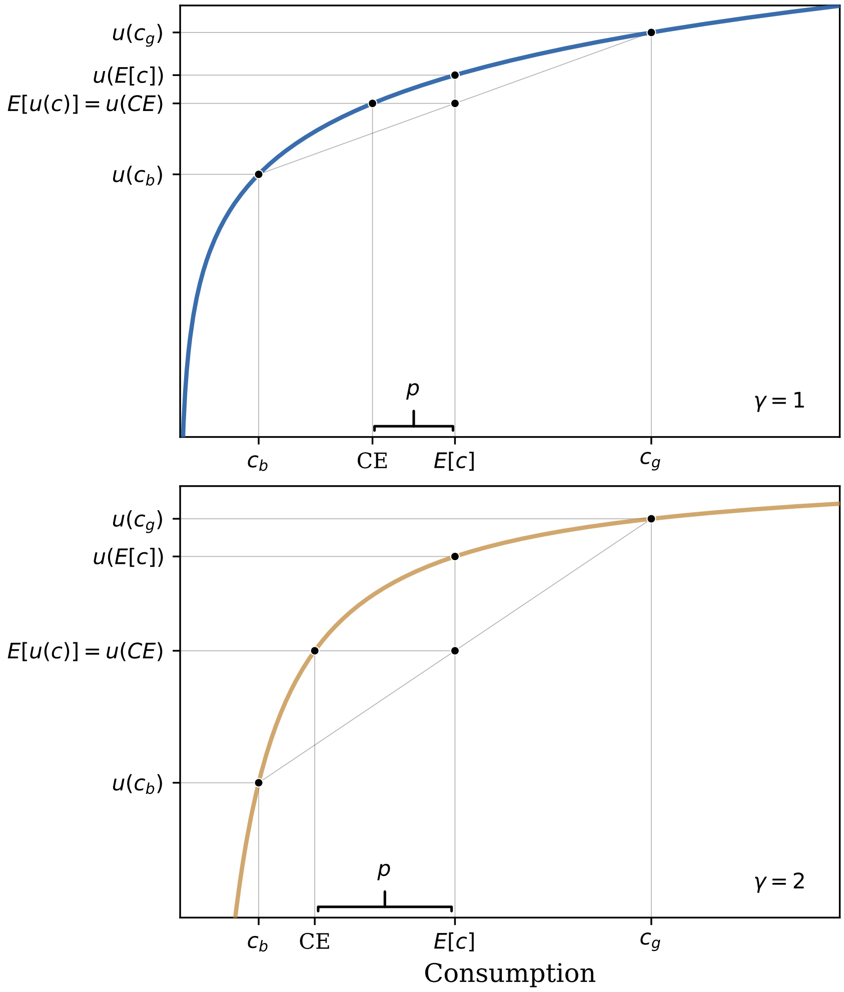


Figure 3.2: Certainty equivalent and risk premium p for different RRA values (top: $\gamma = 1$, bottom: $\gamma = 2$). Both individuals face the same gamble, but the more risk-averse person (with $\gamma = 2$) has a higher risk premium!

Clearly, risk aversion is related to the curvature of the utility function. As an exercise, you can try to draw this graph for a risk-neutral individual who has by definition a linear utility function (and therefore no curvature). You will find that for such an individual the risk premium is zero![3]

[3]If $u(\bullet)$ is linear, it can be interchanged with the expectations operator, so we have $\mathbb{E}[u(c)] = u(\mathbb{E}[c])$

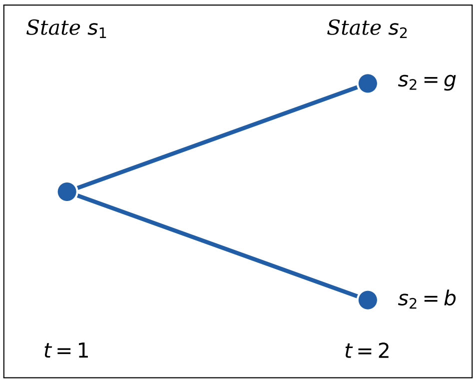


Figure 3.3: Even tree for two periods with uncertainty about state s_2 in the second period.

3.4 Complete markets

We are now in a position to solve our first heterogeneous-agent model with idiosyncratic uncertainty and complete markets. By complete markets we mean a market structure where the household can perfectly insure against idiosyncratic risk by purchasing or selling *contingent bonds*, i.e., bonds that pay off if a specific state of the world is realised tomorrow. If there are no restrictions on the trade of such bonds (e.g., borrowing constraints), we say that markets are complete.

For example, imagine the event tree illustrated in Figure 3.3. In the first period, the state s_1 is observed *before* households make any decision, and is therefore non-stochastic. However, in period 2 there are two possible states of the world, $s_2 = b$ or $s_2 = g$ which are uncertain at $t = 1$. In what follows, we allow the household's income realisation to depend on s_2, and thus period-2 income itself will be uncertain.

3.4.1 Decentralised economy

One way to implement complete markets is to allow for unrestricted trade in so-called *Arrow securities* (named after Nobel laureate Kenneth Arrow). These are one-period bonds which pay one unit of consumption next period, but only if a specific state s_2 obtains. In our example, we have two such securities, one for the state $s_2 = b$ and one for $s_2 = g$, each paying

$$\text{payoff}_b(s_2) = \begin{cases} 1 & \text{if } s_2 = b \\ 0 & \text{if } s_2 = g \end{cases} \tag{3.10}$$

$$\text{payoff}_g(s_2) = \begin{cases} 0 & \text{if } s_2 = b \\ 1 & \text{if } s_2 = g \end{cases} \tag{3.11}$$

We denote the period-1 prices of these Arrow bonds by q_b and q_g, respectively. Using this setup, we can now proceed to solve the household problem with income uncertainty.

and hence $CE = \mathbb{E}[c]$.

Example 3.3 (Household optimality conditions). Consider the following two-period consumption-savings problem with uncertain period-2 income and complete markets:

$$\max_{c_1, c_{2b}, c_{2g}, a_b, a_g} \quad u(c_1) + \beta \underbrace{\left[\pi\, u(c_{2b}) + (1-\pi) u(c_{2g})\right]}_{\equiv \mathbb{E} u(c_2)} \tag{3.12}$$

$$\text{s.t.} \quad c_1 + q_b a_b + q_g a_g = y_1 \tag{3.13}$$

$$c_{2b} = a_b + y_{2b} \tag{3.14}$$

$$c_{2g} = a_g + y_{2g} \tag{3.15}$$

where period-2 income is given by

$$y_2 = \begin{cases} y_{2b} & \text{with probability } \pi \\ y_{2g} & \text{with probability } 1-\pi \end{cases} \tag{3.16}$$

The objective is to maximise expected utility $u(c_1) + \beta \mathbb{E} u(c_2)$ subject to three budget constraints. For period 1, (3.13) states that the household chooses consumption c_1 and can additionally purchase quantities a_b and a_g of Arrow securities at prices q_b and q_g, respectively. Each of these securities pays one unit of consumption in its respective state, so in period 2, the household consumes that payoff as well as any realised income as shown in (3.14) and (3.15).

We can proceed as in earlier units and consolidate all three budget constraints into a single present-value lifetime budget constraint (LTBC). Solving (3.14) and (3.15) for assets and plugging into (3.13), we have that

$$c_1 + q_b (c_{2b} - y_{2b}) + q_g (c_{2g} - y_{2g}) = y_1 \; .$$

Collecting consumption terms on the left-hand side, we obtain the LTBC

$$c_1 + q_b c_{2b} + q_g c_{2g} = y_1 + q_b y_{2b} + q_g y_{2g} \; . \tag{3.17}$$

Note that there is a direct way to arrive at the budget constraint in the complete markets environment. Because the household can trade Arrow securities for all idiosyncratic states of the world, it can outright purchase consumption claims for both possible states in period 2 at prices q_b and q_g. Since we have normalised the price of period-1 consumption to one, the total cost of lifetime consumption is the left-hand side of (3.17). At the same time, the household can *sell* its claims to income y_{2b} and y_{2g} in period 2. Together with y_1, the value of its lifetime income is represented by the right-hand side of (3.17).

We can now set up the Lagrangian with (3.17) as the only constraint:

$$\begin{aligned} \mathcal{L} = u(c_1) + \beta \left[\pi\, u(c_{2b}) + (1-\pi) u(c_{2g})\right] \\ + \lambda \left[y_1 + q_b y_{2b} + q_g y_{2g} - c_1 - q_b c_{2b} - q_g c_{2g}\right] \end{aligned}$$

The first-order conditions for c_1, c_{2b} and c_{2g} are given by

$$\frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda = 0 \tag{3.18}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2b}} = \beta\pi u'(c_{2b}) - \lambda q_b = 0 \tag{3.19}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2g}} = \beta(1-\pi)u'(c_{2g}) - \lambda q_g = 0 \tag{3.20}$$

Since the household can trade in *two* assets (the two Arrow securities), it now has *two* Euler equations (in general, we'll have one Euler equation for each intertemporal choice). The Euler equation for the Arrow bond contingent on $s_2 = b$ can be obtained by combining (3.18) and (3.19) and eliminating λ,

$$u'(c_1)q_b = \beta\pi u'(c_{2b}) \tag{3.21}$$

while the other one follows from (3.18) and (3.20):

$$u'(c_1)q_g = \beta(1-\pi)u'(c_{2g}) \tag{3.22}$$

■

In the previous example, we solved the household's problem for a generic utility function $u(\bullet)$. Next, we derive the optimal consumption rules under the assumption of log preferences.

Example 3.4 (Household problem with log preferences). Continuing with Example 3.3, we now assume that the household has log preferences. The solution with general CRRA works in an analogous fashion, but the expressions are substantially more complicated.

Imposing $u'(c) = \frac{1}{c}$, the Euler equations in (3.21) and (3.22) become

$$\frac{1}{c_1}q_b = \beta\pi\frac{1}{c_{2b}}$$

$$\frac{1}{c_1}q_g = \beta(1-\pi)\frac{1}{c_{2g}}$$

which implies that

$$c_{2b} = \beta\pi\frac{1}{q_b}c_1 \tag{3.23}$$

$$c_{2g} = \beta(1-\pi)\frac{1}{q_g}c_1 \tag{3.24}$$

Denote the value of lifetime income by $\overline{y}$,

$$\overline{y} \equiv y_1 + q_b y_{2b} + q_g y_{2g} \tag{3.25}$$

so that we can rewrite the budget constraint (3.17) as

$$c_1 + q_b c_{2b} + q_g c_{2g} = \overline{y}\ .$$

Substituting for c_{2b} and c_{2g} using (3.23) and (3.24) and solving for c_1, we find

$$\begin{aligned}
c_1 + q_b \beta\pi \frac{1}{q_b} c_1 + q_g \beta(1-\pi)\frac{1}{q_g} c_1 &= \overline{y} \\
c_1 \Big[1 + \beta\pi + \beta(1-\pi)\Big] &= \overline{y} \\
\implies \quad c_1 &= \frac{1}{1+\beta}\overline{y} \qquad (3.26)
\end{aligned}$$

The last expression should look familiar since it is the same as in the case of no uncertainty and log preferences: in the first period, the household consumes a fraction $\frac{1}{1+\beta}$ of the present value of lifetime income.

Plugging in the expression for c_1 into (3.23) and (3.24) yields the optimal consumption levels in period 2 for the states b and g:

$$c_{2b} = \beta\pi \frac{1}{q_b} c_1 = \frac{\beta}{1+\beta}\frac{\pi}{q_b}\overline{y} \qquad (3.27)$$

$$c_{2g} = \beta(1-\pi) \frac{1}{q_g} c_1 = \frac{\beta}{1+\beta}\frac{1-\pi}{q_g}\overline{y} \qquad (3.28)$$

These again don't look very different from what we found in the case of no uncertainty. In fact, if $q_b = \frac{\pi}{1+r}$ and $q_g = \frac{1-\pi}{1+r}$, they would be identical to what we obtained in unit 1.

■

The above example illustrates how to solve the household's problem in partial equilibrium, taking prices q_b and q_g as given. The following example shows how to work out the equilibrium prices in an economy with two households and CRRA preferences.

Example 3.5 (Bond prices in general equilibrium). Consider an economy with two households, A and B, which face idiosyncratic income risk in period 2. Denote by Y_1, Y_{2b} and Y_{2g} the aggregate endowments in the respective periods and states, i.e.,

$$\begin{aligned}
Y_1 &= y_1^A + y_2^B \\
Y_{2b} &= y_{2b}^A + y_{2b}^B \\
Y_{2g} &= y_{2g}^A + y_{2g}^B
\end{aligned}$$

What are the equilibrium prices q_b and q_g in this economy? There are at least two ways to solve this problem:

1. Use the optimal consumption rules and find the market-clearing price vector. Even with log preferences, this requires a lot of algebra and is left to appendix 3.3.

2. Use insights from the first-order conditions to determine the equilibrium price vector.

In this example, we proceed with the second approach. Recall the first-order conditions from (3.18), (3.19) and (3.20). With CRRA preferences, for household type $i = A, B$ these read

$$\begin{aligned}
\left(c_1^i\right)^{-\gamma} &= \lambda_i \\
\beta\pi\left(c_{2b}^i\right)^{-\gamma} &= \lambda_i q_b \\
\beta(1-\pi)\left(c_{2g}^i\right)^{-\gamma} &= \lambda_i q_g
\end{aligned}$$

Dividing the FOCs for A by those for B, we see that

$$\left(\frac{c_1^A}{c_1^B}\right)^{-\gamma} = \frac{\lambda_A}{\lambda_B}, \qquad \left(\frac{c_{2b}^A}{c_{2b}^B}\right)^{-\gamma} = \frac{\lambda_A}{\lambda_B}, \qquad \left(\frac{c_{2g}^A}{c_{2g}^B}\right)^{-\gamma} = \frac{\lambda_A}{\lambda_B}$$

From these equations we conclude that

$$\frac{c_1^A}{c_1^B} = \frac{c_{2b}^A}{c_{2b}^B} = \frac{c_{2g}^A}{c_{2g}^B} = \left(\frac{\lambda_A}{\lambda_B}\right)^{-\frac{1}{\gamma}} \tag{3.29}$$

Note that the right-hand side is a constant! We don't know the individual consumption levels, but the first-order conditions tell us that relative consumption remains unchanged in *all periods and all states*. It is therefore the case that A and B will consume a *constant* fraction of aggregate income each period. Let α be the fraction consumed by A, so that[4]

$$c_1^A = \alpha Y_1\,, \qquad c_{2b}^A = \alpha Y_{2b}\,, \qquad c_{2g}^A = \alpha Y_{2g}$$

Plugging these expressions into A's Euler equation for Arrow bond b, we have

$$\begin{aligned}
\left(c_1^A\right)^{-\gamma} q_b &= \beta\pi\left(c_{2b}^A\right)^{-\gamma} \\
\left(\alpha Y_1\right)^{-\gamma} q_b &= \beta\pi\left(\alpha Y_{2b}\right)^{-\gamma} \\
\implies \quad q_b &= \beta\pi\left(\frac{Y_{2b}}{Y_1}\right)^{-\gamma}
\end{aligned} \tag{3.30}$$

[4]We can easily derive the expression for α from (3.29): For all t and $s \in \{b, g\}$, B's consumption is related to A's consumption such that $c_{ts}^B = (\lambda_A/\lambda_B)^{1/\gamma} c_{ts}^A$. From the aggregate resource constraint it follows that

$$c_{ts}^A + c_{ts}^B = Y_{ts} \implies c_{ts}^A + (\lambda_A/\lambda_B)^{1/\gamma} c_{ts}^A = Y_{ts} \implies c_{ts}^A = \frac{1}{1+(\lambda_A/\lambda_B)^{1/\gamma}} Y_{ts}\,.$$

We hence have $\alpha \equiv \frac{1}{1+(\lambda_A/\lambda_B)^{1/\gamma}}$.

Similarly, using the Euler equation for Arrow bond g, we get

$$q_g = \beta(1-\pi)\left(\frac{Y_{2g}}{Y_1}\right)^{-\gamma} \tag{3.31}$$

What do we learn from the expressions in (3.30) and (3.31)? As you can see, with complete markets the distribution of income does not matter for market prices, as these are only a function of aggregate income and parameters.

■

Now that we computed the equilibrium Arrow bond prices q_b and q_g, we return to the household's problem and solve for the optimal allocation.

Example 3.6 (Equilibrium with symmetric income shocks). Continuing with Example 3.4 and Example 3.5, we now impose that households A and B face symmetric income risk as shown in Table 3.1. Both have identical income $y_1^A = y_1^B = y_1$ in period 1, with aggregate income being $Y_1 = y_1^A + y_1^B = 2 \cdot y_1$.

Household	Income in $t=1$	Income in $t=2$	
		State b (prob. π)	State g (prob. $1-\pi$)
A	y_1	$y_2 - \epsilon$	$y_2 + \epsilon$
B	y_1	$y_2 + \epsilon$	$y_2 - \epsilon$
Aggregate	$Y_1 = 2y_1$	$Y_2 = 2y_2$	$Y_2 = 2y_2$

Table 3.1: Income in economy from Example 3.6.

In the second period, household A can have a good or bad income realisation,

$$y_2^A = \begin{cases} y_{2b}^A = y_2 - \epsilon & \text{with probability } \pi \\ y_{2g}^A = y_2 + \epsilon & \text{with probability } 1-\pi \end{cases} \tag{3.32}$$

where $0 < \epsilon < y_2$. For household B, the income realisations are exactly flipped so that in the aggregate, $Y_2 = 2 \cdot y_2$ with certainty (for household B the labels "good" and "bad" are confusing, but we choose to label the economy from A's perspective).

Let's now use the equilibrium prices we found earlier. With $\gamma = 1$, (3.30) and (3.31) become

$$q_b = \beta\pi\frac{Y_1}{Y_2}$$
$$q_g = \beta(1-\pi)\frac{Y_1}{Y_2}$$

and thus the lifetime income for either type $i = A, B$ is

$$\begin{aligned}
\overline{y}^i &= y_1 + q_b y^i_{2b} + q_g y^i_{2g} \\
&= y_1 + \beta\pi\frac{Y_1}{Y_2}y^i_{2b} + \beta(1-\pi)\frac{Y_1}{Y_2}y^i_{2g} \\
&= y_1 + \beta\frac{Y_1}{Y_2}\underbrace{\left[\pi y^i_{2b} + (1-\pi)y^i_{2g}\right]}_{\mathbb{E}y^i_{2s}}
\end{aligned}$$

To simplify the remainder of this example, we now impose that $\pi = \frac{1}{2}$ and hence

$$\mathbb{E}y^A_{2s} = \pi y^A_{2b} + (1-\pi)y^A_{2g} = \frac{1}{2}(y_2 - \epsilon) + \frac{1}{2}(y_2 + \epsilon) = y_2$$

and analogously for type B. Consequently, we have

$$\overline{y}^i = y_1 + \beta\frac{Y_1}{Y_2}y_2 \, ,$$

and because $Y_t = 2 \cdot y_t$ for $t = 1, 2$,

$$\overline{y}^i = \frac{Y_1}{2} + \beta\frac{Y_1}{Y_2}\frac{Y_2}{2} = (1+\beta)\frac{1}{2}Y_1 \, .$$

Plugging this into (3.26), consumption in period 1 for each household is

$$c^i_1 = \frac{1}{1+\beta}\overline{y}^i = \frac{1}{2}Y_1$$

Lastly, for period 2, from (3.27) and (3.28) it follows that

$$\begin{aligned}
c^i_{2b} &= \frac{\beta}{1+\beta}\frac{\pi}{q_b}\overline{y}^i = \frac{1}{1+\beta}\frac{Y_2}{Y_1}\overline{y}^i = \frac{1}{1+\beta}(1+\beta)\frac{1}{2}Y_2 = \frac{1}{2}Y_2 \\
c^i_{2g} &= \frac{\beta}{1+\beta}\frac{1-\pi}{q_g}\overline{y}^i = \frac{1}{1+\beta}\frac{Y_2}{Y_1}\overline{y}^i = \frac{1}{1+\beta}(1+\beta)\frac{1}{2}Y_2 = \frac{1}{2}Y_2
\end{aligned}$$

Consequently, each household consumes exactly half the aggregate endowment in each period. The intuition behind this result is that both households are ex ante identical: with $\pi = \frac{1}{2}$, each of them was equally likely to get a good or bad draw, hence they should intuitively receive the same share of aggregate income if they can perfectly hedge against idiosyncratic income risk. Consequently, consumption is independent of whether the household turned out to be the lucky or unlucky ex post!

■

3.4.2 Centralised economy

Instead of working through the decentralised equilibrium, we can equivalently solve the social planner's problem. To see this, recall the first fundamental theorem of welfare economics, which loosely speaking states that a decentralised equilibrium with *complete markets*, complete information and perfect competition will be Pareto optimal. All of these criteria are satisfied in our setting, so we could just as well solve for the allocation using the planner's problem, which we do next.

Example 3.7 (Planner's solution). Consider the economy with two agents, A and B, from Example 3.5. We allow for arbitrary income realisations for A and B in period $t = 1, 2$ and state $s = b, g$, and define aggregate endowments as the sum of these,

$$\begin{aligned} Y_1 &= y_1^A + y_2^B \\ Y_{2b} &= y_{2b}^A + y_{2b}^B \\ Y_{2g} &= y_{2g}^A + y_{2g}^B \end{aligned}$$

To set up the planner's problem, we need to assign so-called *Pareto weights* which the planner attaches to each household in the economy. Denoting these weights by θ_i, the planner solves[5]

$$\max_{\left(c_1^i, c_{2b}^i, c_{2g}^i\right)_{i=A,B}} \quad \sum_{i=A,B} \theta_i \left\{ u(c_1^i) + \beta \left[\pi \, u\left(c_{2b}^i\right) + (1-\pi) u\left(c_{2g}^i\right) \right] \right\} \tag{3.33}$$

$$\text{s.t.} \quad \sum_{i=A,B} c_1^i = Y_1 \tag{3.34}$$

$$\sum_{i=A,B} c_{2b}^i = Y_{2b} \tag{3.35}$$

$$\sum_{i=A,B} c_{2g}^i = Y_{2g} \tag{3.36}$$

where $i = A, B$ indexes households A and B.

Let's briefly review the elements of a social planner's problem: The planner maximises a (weighted) sum of utilities of individual households given in (3.33) subject to the aggregate resource constraints (3.34), (3.35) and (3.36). For the planner it is not relevant how endowments are distributed within periods since the planner pools all resources, so it must only observe the *aggregate* resource constraints. Moreover, the planner does not solve for household-specific purchases of Arrow bonds as we did earlier but instead directly allocates consumption to each household in each period and state.

[5] In general, if we want to replicate the decentralised allocation, these weights will depend on households' endowments. The endowments in Example 3.5 are ex ante identical for both households, so their Pareto weights will also be identical.

The Lagrangian for the centralised problem is

$$\mathcal{L} = \sum_{i=A,B} \theta_i \left\{ u(c_1^i) + \beta \left[\pi\, u\left(c_{2b}^i\right) + (1-\pi) u\left(c_{2g}^i\right) \right] \right\}$$
$$+ \lambda_1 \left[Y_1 - \sum_{i=A,B} c_1^i \right] + \lambda_b \left[Y_{2b} - \sum_{i=A,B} c_{2b}^i \right] + \lambda_g \left[Y_{2g} - \sum_{i=A,B} c_{2g}^i \right]$$

and the first-order conditions for c_1^i, c_{2b}^i and c_{2g}^i are

$$\frac{\partial \mathcal{L}}{\partial c_1^i} = \theta_i u'(c_1^i) - \lambda_1 = 0 \tag{3.37}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2b}^i} = \theta_i \beta u'(c_{2b}^i) - \lambda_b = 0 \tag{3.38}$$

$$\frac{\partial \mathcal{L}}{\partial c_{2g}^i} = \theta_i \beta u'(c_{2g}^i) - \lambda_g = 0 \tag{3.39}$$

Note that the Lagrange multipliers λ_1, λ_b and λ_g are the same for all households, hence from (3.37) we have

$$\left.\begin{aligned} \theta_A u'(c_1^A) &= \lambda_1 \\ \theta_B u'(c_1^B) &= \lambda_1 \end{aligned}\right\} \implies \frac{u'(c_1^A)}{u'(c_1^B)} = \frac{\theta_B}{\theta_A}$$

Intuitively, if the planner's weight on A is larger, then the marginal utility of A has to be smaller, i.e., A is allocated a higher consumption level! With CRRA preferences, we can take this one step further to find that

$$\frac{\left(c_1^A\right)^{-\gamma}}{\left(c_1^B\right)^{-\gamma}} = \frac{\theta_B}{\theta_A} \implies \frac{c_1^A}{c_1^B} = \left(\frac{\theta_B}{\theta_A}\right)^{-\frac{1}{\gamma}} \tag{3.40}$$

If you compare this to (3.29) of the decentralised problem, you see that the planner's weight θ_i is the inverse of household i's Lagrange multiplier on the budget constraint, $\theta_i = \lambda_i^{-1}$.

From (3.38) and (3.39), it analogously follows that

$$\frac{c_{2b}^A}{c_{2b}^B} = \frac{c_{2g}^A}{c_{2g}^B} = \left(\frac{\theta_B}{\theta_A}\right)^{-\frac{1}{\gamma}} \tag{3.41}$$

We can now solve for the consumption allocation as a function of parameters and the Pareto weights. For example, for the first period we solve (3.40) for c_1^B and plug it into the resource constraint to see that

$$\begin{aligned} c_1^A + c_1^B &= Y_1 \\ c_1^A + (\theta_B/\theta_A)^{\frac{1}{\gamma}} c_1^A &= Y_1 \\ c_1^A \left[1 + (\theta_B/\theta_A)^{\frac{1}{\gamma}}\right] &= Y_1 \\ \implies c_1^A &= \frac{1}{1 + (\theta_B/\theta_A)^{\frac{1}{\gamma}}} Y_1 \end{aligned}$$

From (3.41) we see that this is in fact true for any period t and each state $s \in \{b, g\}$, hence

$$c_{st}^A = \frac{1}{1 + (\theta_B/\theta_A)^{\frac{1}{\gamma}}} Y_{st}$$

We can verify that whenever A is assigned a higher relative weight θ_A/θ_B, it will be allocated more consumption in each period and state. ■

3.5 Main takeaways

We studied how to characterise the degree of an agent's risk aversion, and we concluded that:

1. More risk-averse agents demand a smaller certainty equivalent, i.e., they accept a smaller *certain* amount instead of a gamble.
2. More risk-averse agents demand a higher risk premium.
3. Risk aversion is connected to the curvature of the utility function. For CRRA preferences, a higher relative risk aversion parameter γ leads to more risk-averse agents.

Moreover, we examined how *idiosyncratic* income risk affects consumption choices in complete markets models.

1. We found that with complete markets, households can perfectly insure against idiosyncratic risk ex ante.
2. Households' allocations and welfare therefore do not depend on their ex post income realisation, i.e., on whether they were lucky or not, but only on aggregate outcomes.
3. Allocations with complete markets are Pareto optimal. We can therefore find an equilibrium by solving either the decentralised or the social planner's problem with appropriate Pareto weights.

Appendix 3.1: Arrow-Pratt coefficient of relative risk aversion

In the main text, we developed the intuition for how risk aversion relates to a consumer's willingness to accept gambles. In this section, we explore the exact meaning of the relative risk aversion parameter.

Risk aversion is related to curvature, but it is inconvenient to characterise it directly in terms of a utility function's second derivative since we can apply arbitrary strictly monotonic transformations to $u(\bullet)$ without changing the underlying risk attitudes, whereas the derivative changes. In economics and finance, we instead use the *Arrow-Pratt coefficient of relative risk aversion* (due to Pratt (1964) and Arrow (1965)) which can be obtained as follows:

Imagine a one-period setting where an individual has initial assets a and is offered the following gamble: with equal probability he or she either wins or loses an amount ϵ with $0 < \epsilon < a$. Because there is no reason to save, the agent chooses to consume all its resources, so the expected utility is

$$\mathbb{E}\left[\, u \,\right] = \frac{1}{2}u(a-\epsilon) + \frac{1}{2}u(a+\epsilon)$$

How much would the individual be willing to pay to avoid this gamble and instead receive a certain amount? Clearly, the individual would be willing to increase the amount p paid until the expected utilities of both scenarios are the same, i.e.,

$$u(a-p) = \frac{1}{2}u(a-\epsilon) + \frac{1}{2}u(a+\epsilon) \tag{3.42}$$

Note that p here is identical to the risk premium from the main text. If we want to solve for p without imposing a particular utility function, we need to approximate the left- and right-hand sides of (3.42) to get p and ϵ outside of $u(\bullet)$ using a method called Taylor series approximation (see appendix 3.2 for details).

Definition 3.1 (Taylor series approximation). Let $f(\bullet)$ be a differentiable function. The first-order Taylor series approximation around a point x_0 is given by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

The second-order Taylor series approximation at x_0 is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

Start with the left-hand side of (3.42): using a first-order approximation around the value of initial assets a, we have

$$u(a-p) \approx u(a) + u'(a)\big((a-p) - a\big) = u(a) - p \cdot u'(a) \tag{3.43}$$

Next, we compute the second-order approximation of the right-hand side of (3.42):[6]

$$\begin{aligned}
\frac{1}{2}\Big[u(a-\epsilon) + u(a+\epsilon)\Big] &\approx u(a) + \frac{1}{2}\Big[u'(a)\big((a-\epsilon) - a\big) + u'(a)\big((a+\epsilon) - a\big)\Big] \\
&\quad + \frac{1}{2}\cdot\frac{1}{2}\Big[u''(a)\big((a-\epsilon) - a\big)^2 + u''(a)\big((a+\epsilon) - a\big)^2\Big] \\
&= u(a) + \frac{1}{2}\epsilon^2 u''(a)
\end{aligned} \tag{3.44}$$

[6]You can try to compute the first-order approximation yourself. You will find that ϵ drops out of the approximate expression, which is not very useful if we aim to relate p to the riskiness of the gamble which depends on ϵ.

Equation (3.42) is therefore approximated by setting (3.43) equal to (3.44):

$$u(a) - p \cdot u'(a) = u(a) + \frac{1}{2}\epsilon^2 u''(a)$$

We can solve this expression for p to find that

$$p = -\frac{1}{2}\epsilon^2 \frac{u''(a)}{u'(a)}$$

The willingness to pay p to avoid the gamble is consequently a function of two terms:

1. the variance of the gamble, ϵ^2 (compare this to Example 3.1 to convince yourself that this indeed is the variance)
2. The *absolute risk aversion* $-u''(a)/u'(a)$. Because our utility function is assumed to be concave, this term is in fact positive.

To arrive at the RRA coefficient we were originally interested in, we need to express the willingness to avoid the gamble relative to the initial asset level a:

$$\frac{p}{a} = \frac{1}{2}\left(\frac{\epsilon}{a}\right)^2 \left(-\frac{u''(a)}{u'(a)}a\right) = \frac{1}{2}\underbrace{\left(\frac{\epsilon}{a}\right)^2}_{\text{Var.}} \underbrace{\left(-\frac{\partial u'(a)}{\partial a}\frac{a}{u'(a)}\right)}_{\text{RRA}} \tag{3.45}$$

Note that all quantities are rescaled by a, so everything is expressed relative to a: the individual is willing to pay p/a per unit, and the per-unit variance of the gamble is given by $(\epsilon/a)^2$. Equation (3.45) thus tells us that willingness to pay per unit variance is proportional to the relative risk aversion. Moreover, the RRA is defined as the *elasticity* of the marginal utility $u'(a)$ with respect to wealth a, i.e., it tells us the relative change in marginal utility as assets change by 1%.[7]

Example 3.8 (Relative risk aversion with CRRA preferences). As the name implies, the relative risk aversion of CRRA preferences is constant and consequently independent of assets, income or consumption. To see this, compute the first two derivatives of

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

which are given by

$$u'(a) = c^{-\gamma} \qquad\qquad u''(c) = -\gamma c^{-\gamma-1}$$

The RRA coefficient is

$$-\frac{u''(c)c}{u'(c)} = -\frac{-\gamma c^{-\gamma-1}c}{c^{-\gamma}} = \gamma$$

and therefore is indeed given by the parameter γ. One implication of constant relative risk aversion is that if households can choose to save in riskless and risky assets, the share invested in risky assets is independent of wealth (at least in models without labour income). ■

[7] Recall that the elasticity of a function $f(x)$ with respect to x is defined as $\frac{\partial f(x)}{\partial x}\frac{x}{f(x)}$.

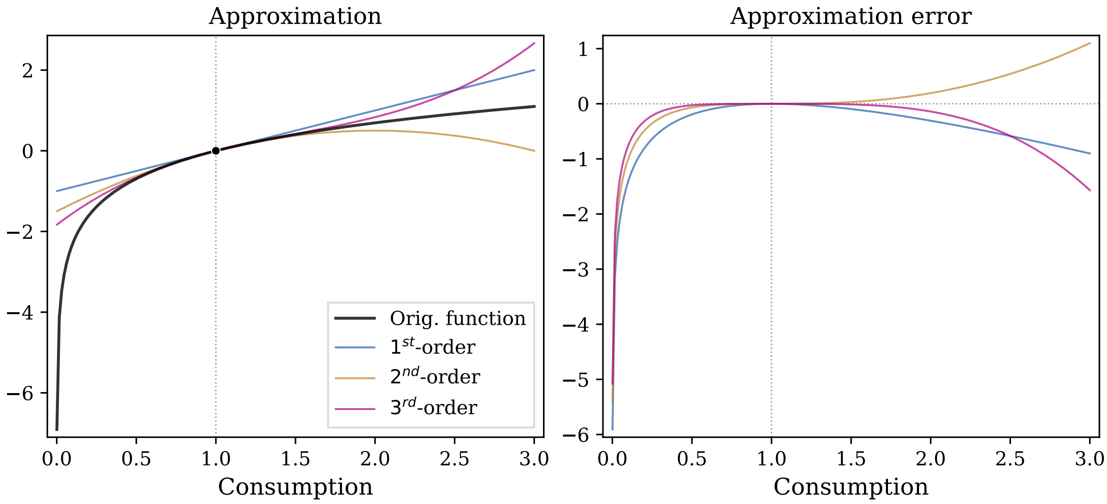


Figure 3.4: Taylor series approximations of log utility $u(c) = \log(c)$ around the point $c = 1$. Left panel shows function values, right panel the approximation error.

Appendix 3.2: Taylor series approximation

We saw the 1^{st}- and 2^{nd}-order Taylor series approximations in Definition 3.1. It is possible to include higher-order terms that make the approximation more precise in a small neighbourhood around the point at which the function is approximated. The approximation represents the function as a polynomial of the desired degree.

Figure 3.4 shows the first three approximations of log preferences $u(c) = \log(c)$ around the point $c = 1$. As you can see, the approximation error close to $c = 1$ tends to be smaller for higher-order polynomials, but on the other hand, these tend to perform *worse* farther away from the point at which we approximate the function!

Appendix 3.3: Finding prices via market clearing

In this section, we illustrate an alternative way to pin down equilibrium prices for the economy in Example 3.4. We require equilibrium in three markets:

1. Period-1 consumption
2. Period-2 consumption in state b
3. Period-2 consumption in state g

We choose to clear the first two of these, and the third one clears by Walras' Law.

Start with period-1 consumption: plugging the optimal consumption rule (3.26) into the aggregate resource constraint, we have

$$
\begin{aligned}
c_1^A + c_2^B &= Y_1 \\
\frac{1}{1+\beta}\overline{y}^A + \frac{1}{1+\beta}\overline{y}^B &= Y_1
\end{aligned}
$$

Next, we substitute in the expressions for lifetime income from (3.25):

$$
\begin{aligned}
\frac{1}{1+\beta}\left[y_1^A + q_b y_{2b}^A + q_g y_{2g}^A + y_1^B + q_b y_{2b}^B + q_g y_{2g}^B\right] &= Y_1 \\
\frac{1}{1+\beta}\left[y_1^A + y_1^B + q_b\left(y_{2b}^A + y_{2b}^B\right) + q_g\left(y_{2g}^A + y_{2g}^B\right)\right] &= Y_1 \\
\frac{1}{1+\beta}\left[Y_1 + q_b Y_{2b} + q_g Y_{2g}\right] &= Y_1 \\
q_b Y_{2b} + q_g Y_{2g} &= \beta Y_1
\end{aligned}
$$

Solving for q_g yields

$$
q_g = \frac{\beta Y_1 - q_b Y_{2b}}{Y_{2g}} \,. \tag{3.46}
$$

We proceed in exactly the same fashion for consumption in period 2 in state b. Plugging in optimal consumption (3.27) into the aggregate resource constraint, we have

$$
\begin{aligned}
c_{2b}^A + c_{2b}^B &= Y_{2b} \\
\frac{1}{1+\beta}\left[\beta\frac{\pi}{q_b}\overline{y}^A + \beta\frac{\pi}{q_b}\overline{y}^B\right] &= Y_{2b} \\
\frac{\beta}{1+\beta}\frac{\pi}{q_b}\left[y_1^A + q_b y_{2b}^A + q_g y_{2g}^A + y_1^B + q_b y_{2b}^B + q_g y_{2g}^B\right] &= Y_{2b} \\
\beta\pi\left[Y_1 + q_b Y_{2b} + q_g Y_{2g}\right] &= (1+\beta) q_b Y_{2b} \\
\beta\pi q_g Y_{2g} &= (1+\beta-\pi\beta)\, q_b Y_{2b} - \beta\pi Y_1
\end{aligned}
$$

Solving for q_g, we get

$$
q_g = \frac{(1+\beta-\pi\beta)\, q_b Y_{2b} - \beta\pi Y_1}{\beta\pi Y_{2g}} \,. \tag{3.47}
$$

We can now equate (3.46) and (3.47) and solve for q_b:

$$
\begin{aligned}
\frac{\beta Y_1 - q_b Y_{2b}}{Y_{2g}} &= \frac{(1+\beta-\pi\beta)\, q_b Y_{2b} - \beta\pi Y_1}{\beta\pi Y_{2g}} \\
\beta^2\pi Y_1 - \beta\pi q_b Y_{2b} &= (1+\beta-\pi\beta)\, q_b Y_{2b} - \beta\pi Y_1 \\
\beta\pi(1+\beta) Y_1 &= (1+\beta) q_b Y_{2b} \\
\implies q_b &= \beta\pi\frac{Y_1}{Y_{2b}}
\end{aligned}
$$

which is exactly what we found in (3.30) for the case of $\gamma = 1$. Lastly, substituting for q_b in (3.46), we get

$$q_g = \frac{\beta Y_1 - \beta \pi Y_1}{Y_{2g}} = \beta(1-\pi)\frac{Y_1}{Y_{2g}}$$

which again is identical to what we found in (3.31) for $\gamma = 1$.

References

Arrow, Kenneth Joseph (1965). *Aspects of the theory of risk-bearing*. Helsinki: Yrjö Jahnssonin Säätiö.

Pratt, John W. (1964). "Risk Aversion in the Small and in the Large". In: *Econometrica* 32.1/2, pp. 122–136.

Exercises

Exercise 3.1 (Risk-neutral and risk-loving preferences). In section 3.3.1, we discussed the certainty equivalent and risk premium for an agent with risk-averse preferences.

(a) Imagine instead that the agent is risk-neutral, i.e., the utility function is linear, $u(c) = c$. Create a graph analogous to Figure 3.2 which illustrates the case of risk-neutral preferences. Clearly indicate the certainty equivalent!

(b) Now consider a risk-loving individual with preferences given by $u(c) = c^2$. Again, create a graph showing the certainty equivalent and risk premium!

Exercise 3.2 (Risk-free bond). Assume a two-period consumption-savings problem with two possible states b and g in the second period as in Example 3.5 where an agent can trade in Arrow bonds with payoffs given in (3.10) and (3.11). Recall the pricing equations for the associated Arrow bonds we derived in equations (3.30) and (3.31), repeated here for convenience:

$$q_b = \beta\pi \left(\frac{Y_{2b}}{Y_1}\right)^{-\gamma}$$
$$q_g = \beta(1-\pi) \left(\frac{Y_{2g}}{Y_1}\right)^{-\gamma}$$

(a) Consider a risk-free bond which pays one unit of consumption in period 2 irrespective of which state realises:

$$\text{payoff}(s_2) = \begin{cases} 1 & \text{if } s_2 = b \\ 1 & \text{if } s_2 = g \end{cases}$$

Derive the equilibrium price q of this risk-free bond! *Hint:* Create a portfolio with the same payoff and compute its price.

(b) What is the risk-free interest rate (i.e., the risk-free bond return) as a function of q? To answer this question, recall that the one-period gross return R_t of any asset with price p and dividend (or coupon) d is given by

$$R_t = \frac{p_{t+1} + d_{t+1}}{p_t}$$

The return is thus the price tomorrow plus any additional payments (coupons, dividends, etc.) divided by the price that has to be paid to purchase the asset today.

(c) Assume that in period two, the risk-averse agent gets income which is either y_{2b} or y_{2g} with $y_{2b} < y_{2g}$. Would such an individual prefer to invest only in the risk-free bond over a portfolio of Arrow bonds b and g?

Exercise 3.3 (Complete markets with a representative agent). Consider a two-period representative-agent (RA) economy with uncertainty about the endowment realisation in period 2, which can be either Y_{2b} or Y_{2g}:

$$Y_2 = \begin{cases} Y_{2b} & \text{with probability } \pi \\ Y_{2g} & \text{with probability } 1-\pi \end{cases} \tag{E.1}$$

Income in the first period is deterministic and given by Y_1.

The representative agent can trade in Arrow bonds which are contingent on states b and g, and decides on consumption for period 1 as well as for states b and g in period 2. The maximisation problem reads

$$\begin{aligned} \max_{C_1, C_{2b}, C_{2g}, A_b, A_g} \quad & u(C_1) + \beta\Big[\pi u(C_{2b}) + (1-\pi)u(C_{2g})\Big] \\ \text{s.t.} \quad C_1 + q_b A_b + q_g A_g &= Y_1 \\ C_{2b} &= Y_{2b} + A_b \\ C_{2g} &= Y_{2g} + A_g \end{aligned}$$

where A_b and A_g are the quantities of Arrow bonds purchased in the first period.

(a) State the lifetime budget constraint for this problem.

(b) Let λ be the Lagrange multiplier on the lifetime budget constraint and derive the first-order conditions.

(c) Derive the Euler equations for each Arrow bond.

(d) Since we have a representative-agent endowment economy, we know that the RA will consume its entire income each period, therefore $C_{ts} = Y_{ts}$ for all periods t and states s. Use this insight and the Euler equations from earlier to find an expression for Arrow bond prices as a function of parameters and aggregate income. From now on, assume that the RA has CRRA preferences with relative risk-aversion parameter γ.

(e) How do these expressions compare to the prices we found in Example 3.5 for the heterogeneous-agent case? Can you draw any conclusions about aggregation of the economy in Example 3.5, i.e., whether it can be modelled using a representative agent?

4 Incomplete markets

4.1 Introduction

In the previous unit, we studied consumption-savings decisions under uncertainty in a setting of complete markets. We concluded that households could insure against all idiosyncratic risk, and consequently their choices looked very much like those in an environment without uncertainty.

While in reality there is some state-contingent insurance which limits the risk associated with specific events (think of unemployment or disability insurance), we don't think that household can insure against every possible risk they face. In this unit, we therefore analyse an environment which imposes the assumption that households can only trade assets that allow them to smooth consumption over time (as in the deterministic consumption-savings problem), but not across different income realisations.

We do this in two settings with very different implications: First, in the certainty equivalence model, we will see that household choices are almost identical to the deterministic case. Conversely, in the precautionary savings model, households respond to changes in income risk which gives rise to optimal choices and equilibria that deviate from the deterministic environment.

4.2 Two-period problem with incomplete markets

Consider the following two-period consumption-savings problem with uncertain period-2 income:

$$\max_{c_1, c_2, a_2} \; u(c_1) + \beta \mathbb{E} u(c_2) \tag{4.1}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1 \tag{4.2}$$

$$c_2 = (1+r)a_2 + y_2 \tag{4.3}$$

$$a_2 \geq -\frac{y_{min}}{1+r} \tag{4.4}$$

$$y_2 \;\text{ stochastic with } y_2 \geq y_{min}$$

It is instructive to compare this to the complete markets setting we discussed in the previous unit. In both cases, the household maximises expected utility (4.1), but the budget constraints are very different. In (4.2), the household can no longer trade contingent bonds but instead can only save or borrow in a risk-free asset which is independent of the income realisation next period.

Moreover, since we do not allow the household to die in debt, we need to impose the borrowing constraint (4.4) which is called the *natural borrowing limit*. Assuming that income is stochastic and that the lowest possible income realisation is $y_{min} \geq 0$, it states that the household can borrow only as much as it can repay in the second period with certainty. To see this, assume that the household chooses to borrow up to the limit so that $a_2 = -y_{min}/(1+r)$. Including interest, it will have to repay

$$(1+r)a_2 = (1+r)\frac{y_{min}}{1+r} = y_{min}$$

which is exactly the lowest possible period-2 income. Consequently, with $y_2 \geq y_{min}$, the household will be able to repay its debt with certainty and will consume the remainder of its income.

One way to solve this problem is to eliminate c_1 and c_2 and keep a_2 as the only choice variable. The problem then reads

$$\max_{a_2} \; u(y_1 - a_2) + \beta\mathbb{E}\left[u\big((1+r)a_2 + y_2\big) \right] \tag{4.5}$$

$$\text{s.t.} \quad a_2 \geq -\frac{y_{min}}{1+r} \tag{4.6}$$

$$y_2 \;\; \text{stochastic with } y_2 \geq y_{min}$$

and the corresponding Lagrangian is given by

$$\mathcal{L} = u(y_1 - a_2) + \beta\mathbb{E}\left[u\big((1+r)a_2 + y_2\big) \right] + \lambda\left[a_2 + \frac{y_{min}}{1+r}\right]$$

where $\lambda \geq 0$ is the Lagrange multiplier on the borrowing constraint (4.6). The first-order condition for a_2 is

$$\frac{\partial \mathcal{L}}{\partial a_2} = -u'(y_1 - a_2) + \beta(1+r)\mathbb{E}\left[u'\big((1+r)a_2 + y_2\big) \right] + \lambda = 0$$

Assuming that the the borrowing constraint is not binding and hence $\lambda = 0$, the first-order condition can be rearranged to yield the Euler equation:

$$u'(\underbrace{y_1 - a_2}_{c_1}) = \beta(1+r)\mathbb{E}u'\big(\underbrace{(1+r)a_2 + y_2}_{c_2}\big) \tag{4.7}$$

which is almost identical to the deterministic case, except that now we have *expected* marginal utility on the right-hand side.

4.3 The certainty equivalence model

4.3.1 Quadratic utility

Up till now we haven't made any specific assumption about preferences. In this section, we impose that the household has a quadratic utility function given by

$$u(c) = \alpha c - \frac{\delta}{2}c^2 \tag{4.8}$$

with parameters $\alpha > 0$ and $\delta > 0$. Since we haven't encountered these preferences so far, let's spend some time discussing their properties. First, as can be seen from Figure 4.1, the quadratic utility function has a so-called *bliss point* at which utility is maximised. This point is located at $c^* = \frac{\alpha}{\delta}$, so a realistic parametrisation needs to impose values for α

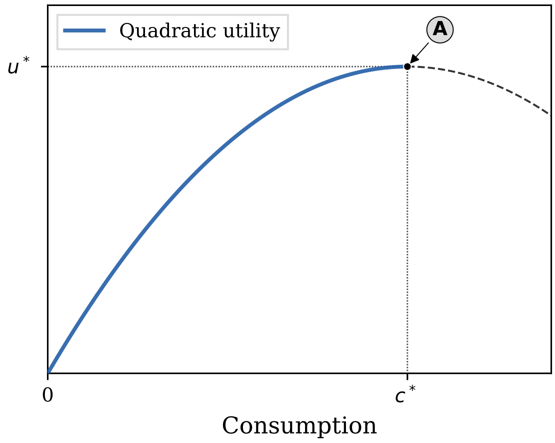


Figure 4.1: Quadratic utility function. (A) shows the bliss point where utility is maximised.

and δ such that any equilibrium consumption is to the left of this point. See appendix 4.2 for details on how these parameters influence the slope and location of the quadratic utility function.

Second, marginal utility is given by

$$u'(c) = \alpha - \delta c \tag{4.9}$$

and is consequently *linear*.[1] This greatly simplifies solving the household problem because it allows us to interchange expectations and the marginal utility function, as we'll see shortly. Linear marginal utility gives rise to what is called *certainty equivalence*: agents make choices which are identical to a setting without uncertainty in which they receive the expected value instead of an uncertain outcome. However, this does not imply risk neutrality! A quadratic utility function is strictly concave, and consequently households are risk averse and demand a risk premium when facing an uncertain gamble, as illustrated in Figure 4.2.[2]

Quadratic preferences are not widely used in modern macroeconomics because of several undesirable properties:

1. As mentioned above, utility is decreasing to the right of the bliss point.

2. The utility function does not satisfy the Inada conditions because utility does not approach $-\infty$ as consumption goes to 0, hence there is no guarantee that the household chooses positive consumption.

[1]Strictly speaking, marginal utility in this case is an *affine* function which is a linear function plus a constant. We ignore this subtle distinction here since it makes no difference for our exposition.

[2]Recall from the previous unit that the risk premium p is the difference between the expected outcome and the certainty equivalent, $p = \mathbb{E}c - CE$.

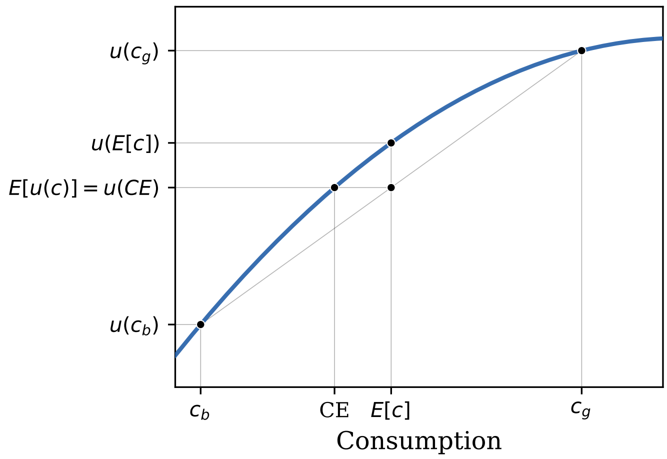


Figure 4.2: Certainty equivalent (CE) with quadratic preferences. The graph shows a situation in which the consumer faces a gamble with potential outcomes c_b and c_g with equal probability. $\mathbb{E}[c]$ is the expected consumption and CE the gamble's certainty equivalent. The risk premium is given by $p = \mathbb{E}[c] - CE$.

3. Quadratic preferences imply an increasing relative risk aversion which is empirically implausible as richer households tend to have riskier financial portfolios.

The only reason to use quadratic preferences is that they are analytically convenient, which we exploit below. Note, however, that quadratic utility is (often implicitly) used in the Capital Asset Pricing Model (CAPM) in finance.

4.3.2 Household problem without uncertainty

It is instructive to solve the deterministic household problem with quadratic preferences so that we can compare it to the scenario with uncertainty. For now, we are therefore back to our standard two-period consumption-savings problem which we have solved repeatedly in the last few units:

$$\max_{c_1, c_2, a_2} \quad u(c_1) + \beta u(c_2) \tag{4.10}$$

$$\begin{aligned}
\text{s.t.} \quad c_1 + a_2 &= y_1 \\
c_2 &= (1+r)a_2 + y_2 \\
c_1 \geq 0,\ c_2 &\geq 0 \qquad (4.11)\\
u(c) &= \alpha c - \frac{\delta}{2} c^2
\end{aligned}$$

The only addition here are the explicit constraints (4.11) on consumption which has to be non-negative, as with quadratic preferences there is nothing that would prevent agents from potentially choosing zero or negative consumption. We will, however, ignore these constraints and assume that they are satisfied at the optimum. The Euler equation for this problem is the usual one,

$$u'(c_1) = \beta(1+r)u'(c_2)\ .$$

Using the marginal utility function from (4.9), it evaluates to

$$\alpha - \delta c_1 = \beta(1+r)\Big[\alpha - \delta c_2\Big]\ .$$

Solving for c_2 yields

$$\begin{aligned}
\alpha - \delta c_1 &= \alpha\beta(1+r) - \delta\beta(1+r)c_2 \\
\alpha\big[1-\beta(1+r)\big] - \delta c_1 &= -\delta\beta(1+r)c_2 \\
\implies \quad c_2 &= \frac{c_1}{\beta(1+r)} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{\beta(1+r)}
\end{aligned} \tag{4.12}$$

To check whether this expression is plausible, we could set $\beta(1+r) = 1$ in which case it reduces to $c_2 = c_1$. In the absence of any incentives to shift consumption intertemporarily, the household will thus consume the same amount in each period.

From here we proceed as in previous units with CRRA preferences. The lifetime budget constraint for this problem is

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

where we can substitute for c_2 using (4.12):

$$c_1 + \frac{1}{1+r}\left[\frac{c_1}{\beta(1+r)} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{\beta(1+r)}\right] = y_1 + \frac{y_2}{1+r}$$

Solving for c_1, we find that

$$\begin{aligned}
c_1\left[1 + \frac{1}{\beta(1+r)^2}\right] &= y_1 + \frac{y_2}{1+r} + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{\beta(1+r)^2} \\
c_1\left[\frac{1+\beta(1+r)^2}{\beta(1+r)^2}\right] &= y_1 + \frac{y_2}{1+r} + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{\beta(1+r)^2} \\
c_1 &= \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\left[y_1 + \frac{y_2}{1+r}\right] + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2}
\end{aligned} \tag{4.13}$$

Lastly, we can find optimal savings by plugging (4.13) into the period-1 budget constraint:

$$\begin{aligned}
a_2 &= y_1 - c_1 \\
&= y_1 - \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\left[y_1 + \frac{y_2}{1+r}\right] - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \\
&= \frac{y_1 - \beta(1+r)^2 y_1 + \beta(1+r)^2 y_1 - \beta(1+r)^2\frac{y_2}{1+r}}{1+\beta(1+r)^2} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \\
&= \frac{y_1 - \beta(1+r)y_2}{1+\beta(1+r)^2} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2}
\end{aligned} \tag{4.14}$$

These expression are quite complex, so let's look at the optimal (c_1, c_2) allocation for specific parameter values.

Example 4.1 (Quadratic preferences without uncertainty). Consider the household problem in (4.10) with optimal period-1 consumption c_1 and savings a_2 given by (4.13) and (4.14), respectively.

Assuming that $\beta = 1$, $y_1 = y_2 = 1$, and the parameters governing quadratic utility $\alpha = 20$ and $\delta = 2$, optimal allocations for three different interest rates are shown in Figure 4.3. As we would expect, for $\beta(1+r) < 1$ in panel (a), the household is impatient relative to the interest rate and thus chooses to borrow ($c_1 > y_1$). Conversely, for $\beta(1+r) > 1$ in panel (c), the household is a saver ($c_1 < y_1$).

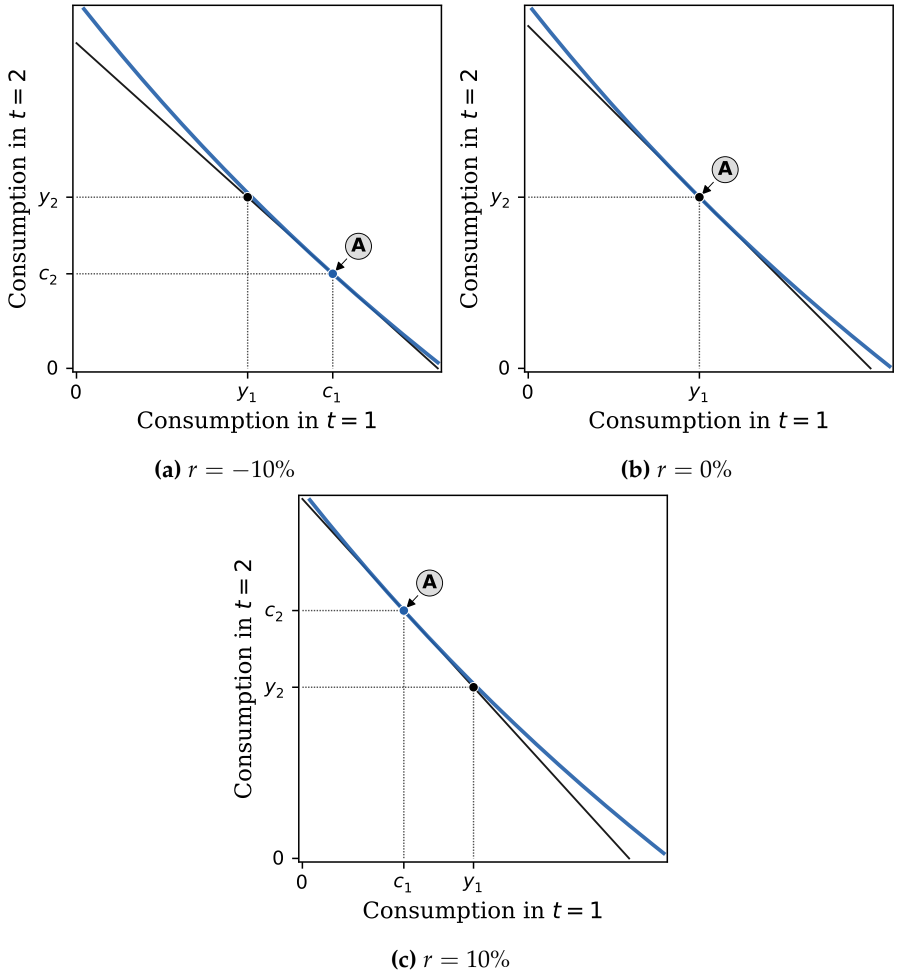


Figure 4.3: Intertemporal consumption choice with quadratic preferences and different interest rates. (A) depicts the optimal allocation (c_1, c_2) and the corresponding indifference curve is represented by the blue line.

■

Example 4.2 (Quadratic preferences with $\beta(1+r) = 1$). Consider the household problem in (4.10) with optimal period-1 consumption c_1 and savings a_2 given by (4.13) and (4.14), respectively.

Assuming that $\beta(1+r) = 1$, the solution to the household problem simplifies considerably. Applying these parameter values to (4.13), period-1 consumption is given by

$$c_1 = \frac{1+r}{2+r}\left[y_1 + \frac{y_2}{1+r}\right]$$

while optimal savings (4.14) become

$$a_2 = \frac{y_1 - y_2}{2+r} \tag{4.15}$$

If we additionally assume that $r = 0$ (and thus $\beta = 1$), we find that

$$c_1 = \frac{1}{2}(y_1 + y_2)$$

so the household consumes exactly half its lifetime income in the first period.

If, on the other hand, we impose $y_1 = y_2$ on top of $\beta(1+r) = 1$, the household chooses not to save anything as (4.15) evaluates to $a_2 = 0$. This case in shown in Figure 4.3 panel (b). ■

4.3.3 Household problem with uncertainty

Now that we learned how to solve the quadratic utility model without uncertainty, we return to our setting with stochastic income. Consider the maximisation problem in (4.5) and assume that the household has quadratic preferences given in (4.8). There is no need to solve the problem from scratch, so we start with the Euler equation in (4.7), assuming that the household is not at the borrowing limit. Using the marginal utility function from (4.9), the Euler equation now reads

$$\begin{aligned} \alpha - \delta c_1 &= \beta(1+r)\mathbb{E}\left[\alpha - \delta c_2\right] \\ &= \alpha\beta(1+r) - \delta\beta(1+r)\mathbb{E}c_2 \end{aligned} \tag{4.16}$$

Let's inspect the second line carefully, since this is at the centre of the certainty equivalence result. For any linear function $f(\bullet)$, we can swap the function and the expectations operator, i.e.,

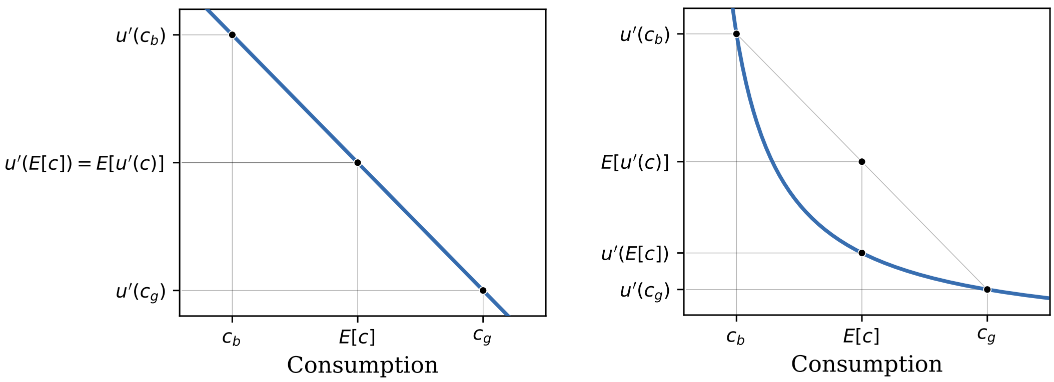
$$\mathbb{E}\left[f(X)\right] = f(\mathbb{E}X) .$$

The linear function here is the marginal utility from (4.9), and that's why

$$\mathbb{E}\left[u'(c_2)\right] = u'\left(\mathbb{E}c_2\right) = \alpha - \delta\mathbb{E}c_2 .$$

Note that this would not be possible with CRRA preferences, as is illustrated in Figure 4.4, since it that case marginal utility $u'(c) = c^{-\gamma}$ is *not* linear and hence

$$\mathbb{E}\left[c_2^{-\gamma}\right] \neq \left(\mathbb{E}c_2\right)^{-\gamma} .$$



(a) Quadratic utility, $u'(c) = \alpha - \delta c$ **(b)** Log utility, $u'(c) = \frac{1}{c}$

Figure 4.4: Marginal utility for quadratic vs. CRRA preferences. The graph shows a situation in which the consumer faces a gamble with potential outcomes c_b and c_g with equal probability and an expected value of $\mathbb{E}[c]$.

One way to approach this problem is to plug in the budget constraints (4.2) and (4.3) into (4.16),

$$\alpha - \delta(y_1 - a_2) = \alpha\beta(1+r) - \delta\beta(1+r)\mathbb{E}[(1+r)a_2 + y_2]$$

To solve for a_2, we first pull a_2 out of the expectations:

$$\alpha - \delta y_1 + \delta a_2 = \alpha\beta(1+r) - \delta\beta(1+r)^2 a_2 - \delta\beta(1+r)\mathbb{E}y_2$$

We can do this because a_2 is chosen by the household and as such is not stochastic! Disentangling the non-stochastic terms from the random y_2 is possible because marginal utility is linear but won't work in general. Collecting a_2 terms on the left-hand side and solving for a_2, we have

$$\begin{aligned} a_2\left[\delta + \delta\beta(1+r)^2\right] &= -\alpha + \alpha\beta(1+r) + \delta y_1 - \delta\beta(1+r)\mathbb{E}y_2 \\ a_2\delta\left[1 + \beta(1+r)^2\right] &= \delta\left[y_1 - \beta(1+r)\mathbb{E}y_2\right] - \alpha(1 - \beta(1+r)) \\ a_2 &= \frac{y_1 - \beta(1+r)\mathbb{E}y_2}{1+\beta(1+r)^2} - \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \end{aligned} \tag{4.17}$$

Lastly, we can use the period-1 budget constraint to find the optimal c_1 under uncertainty:

$$\begin{aligned} c_1 &= y_1 - a_2 \\ &= y_1 - \frac{y_1 - \beta(1+r)\mathbb{E}y_2}{1+\beta(1+r)^2} + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \\ &= \frac{y_1 + \beta(1+r)^2 y_1 - y_1 + \beta(1+r)\mathbb{E}y_2}{1+\beta(1+r)^2} + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \\ &= \frac{\beta(1+r)^2}{1+\beta(1+r)^2}\left[y_1 + \frac{\mathbb{E}y_2}{1+r}\right] + \frac{\alpha}{\delta}\frac{1-\beta(1+r)}{1+\beta(1+r)^2} \end{aligned} \tag{4.18}$$

At this point, we can compare the solution with and without uncertainty. For optimal c_1, we see that (4.13) and (4.18) are identical, except that in the latter y_2 was replaced with its expected value, $\mathbb{E}y_2$. The same is true for savings in the first period, see (4.14) and (4.17). We conclude that with quadratic preferences, the solution under uncertainty is identical to the deterministic one if we replace the income realisation with its expected value, giving rise to what is called *certainty equivalence*. Note, however, that c_2 will *not* be the same in both scenarios in general, unless the realised value of y_2 coincides with its expectation.

The expressions in (4.17) and (4.18) are again hard do interpret, so as before we resort to specific numerical examples.

Example 4.3 (Consumption-savings with quadratic utility). Consider the stochastic income problem (4.5) with quadratic utility, and optimal savings and consumption choices given in (4.17) and (4.18), respectively.

Let $y_1 = \mathbb{E}y_2 = 1$, $y_{min} = 0.5$ (so that the borrowing limit is never binding), $\beta = 1$, and assume that quadratic utility is parametrised by $\alpha = 20$ and $\delta = 2$. Figure 4.5 shows the optimal consumption and savings levels for a range of interest rates $r \in [-0.1, 0.1]$. For $r = 0$, c_1 and the expected consumption level $\mathbb{E}c_2$ coincide, which directly follows from the Euler equation.

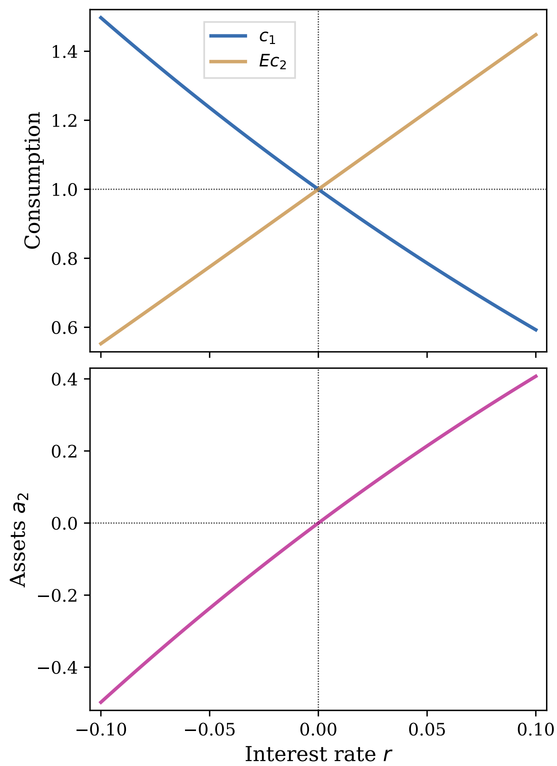


Figure 4.5: Optional consumption and savings with quadratic utility under uncertainty, as in Example 4.3.

■

4.4 Precautionary savings model

In the previous section, we saw that the certainty equivalence solution leads to choices that are no different from the deterministic case. This seems implausible as one would expect that people facing higher risk would increase their *precautionary savings* to insure themselves against adverse outcomes. Indeed, empirical evidence suggests that households with more volatile income have higher savings rates. This, in addition to all the other shortcomings of quadratic utility mentioned earlier, makes the certainty equivalence framework unappealing in modern macroeconomics and household finance.

We therefore return to our standard CRRA framework, and assume that the household solves

$$\begin{aligned} &\max_{c_1, c_2, a_2} \; u(c_1) + \beta \mathbb{E}\left[u(c_2) \right] && (4.19) \\ \text{s.t.} \quad & c_1 + a_2 = y_1 && (4.20) \\ & c_2 = (1+r)a_2 + y_2 && (4.21) \\ & a_2 \geq -\frac{y_{min}}{1+r} && (4.22) \\ & y_2 \text{ stochastic with } y_2 \geq y_{min} \\ & u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases} \end{aligned}$$

The Euler equation in (4.7) for the CRRA case with uncertainty becomes

$$c_1^{-\gamma} = \beta(1+r)\mathbb{E}\left[c_2^{-\gamma} \right] . \tag{4.23}$$

From the above expression it is evident that in general we won't be able to proceed as we did in all earlier cases when we expressed c_2 as a function of c_1 and substituted for c_2 in the lifetime budget constraint. That is not possible here since

$$\mathbb{E}\left[c_2^{-\gamma} \right] \neq (\mathbb{E}c_2)^{-\gamma} .$$

In fact, because marginal utility is a strictly convex function, we have

$$\mathbb{E}\left[c_2^{-\gamma} \right] > (\mathbb{E}c_2)^{-\gamma}$$

which follows from Jensen's inequality and is illustrated in Figure 4.4 panel (b). Compared to the case of certainty equivalence, the right-hand side of the Euler equation (4.23) will be larger and hence by optimality the left-hand side has to increase as well. The only way to increase marginal utility in period 1 is to decrease consumption, which gives rise to precautionary savings in the presence of risk, a mechanism that was absent in the certainty equivalence model!

How can we proceed to solve the problem? The approach we pursued in the previous section was to eliminate consumption and express the Euler equation in terms of savings a_2, which in this case reads

$$(y_1 - a_2)^{-\gamma} = \beta(1+r)\mathbb{E}\left[\left((1+r)a_2 + y_2\right)^{-\gamma} \right] .$$

This is a nonlinear equation in a single unknown a_2 which in general cannot be solved analytically. Even imposing log preferences, our usual approach to fix complicated problems, does not help:

$$\frac{1}{y_1 - a_2} = \beta(1+r)\mathbb{E}\left[\frac{1}{(1+r)a_2 + y_2} \right] \tag{4.24}$$

In the literature, there are several ways to solve such problems:

1. Replace the terms inside the expectation with a higher-order Taylor approximation.
2. Make assumptions on the distribution of consumption in period 2, e.g., that it is log-normal. This is common in finance where consumption is often taken to be exogenous, but is not appealing in macroeconomics.
3. Solve the problem numerically.

These methods are beyond the scope of this course, so we will instead impose sufficiently many assumptions to simplify the problem so that it can be solved analytically.

Example 4.4 (Precautionary savings with log preferences). Consider the household problem in (4.19), and assume that $\gamma = 1$, $\beta = 1$, $y_1 = y$ and y_2 is given by

$$y_2 = \begin{cases} y - \epsilon & \text{with prob. } \frac{1}{2} \\ y + \epsilon & \text{with prob. } \frac{1}{2} \end{cases} \tag{4.25}$$

where $0 < \epsilon < y$. The period-2 income risk is thus symmetric with expectation $\mathbb{E}y_2 = y$. With this parametrisation, we have $y_{min} = y - \epsilon$ and hence the household can borrow up to an amount of

$$a_2 \geq -\frac{y-\epsilon}{1+r}$$

while still being able to repay its debt with certainty.

Under these assumptions, the Euler equation (4.24) becomes

$$\frac{1}{y - a_2} = (1+r)\left[\frac{1}{2}\frac{1}{(1+r)a_2 + y - \epsilon} + \frac{1}{2}\frac{1}{(1+r)a_2 + y + \epsilon} \right] \tag{4.26}$$

It is now possible to extract a_2 from the expectation by noting that

$$\left[(1+r)a_2 + y - \epsilon\right]\left[(1+r)a_2 + y + \epsilon\right] = \left[(1+r)a_2 + y\right]^2 - \epsilon^2$$

Therefore, the expectation in (4.26) can be transformed to a common denominator as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{1}{(1+r)a_2+y-\epsilon}+\frac{1}{2}\frac{1}{(1+r)a_2+y+\epsilon}\\
&\qquad=\frac{1}{2}\frac{(1+r)a_2+y+\epsilon}{\left[(1+r)a_2+y\right]^2-\epsilon^2}+\frac{1}{2}\frac{(1+r)a_2+y-\epsilon}{\left[(1+r)a_2+y\right]^2-\epsilon^2}\\
&\qquad=\frac{(1+r)a_2+y}{\left[(1+r)a_2+y\right]^2-\epsilon^2}
\end{aligned}$$

The Euler equation therefore simplifies to

$$\frac{1}{y-a_2}=(1+r)\frac{(1+r)a_2+y}{\left[(1+r)a_2+y\right]^2-\epsilon^2} \tag{4.27}$$

which is a quadratic equation in a_2. Solving for a_2 is straightforward but involves a lot of algebra, so we leave it to appendix 4.1, where we find that optimal a_2 is given by

$$a_2=-\frac{(2-r)y}{4(1+r)}+\frac{\sqrt{(2+r)^2y^2+8\epsilon^2}}{4(1+r)} \tag{4.28}$$

It can be quite challenging to assess whether such a complicated expression is correct, so the usual approach is to verify some basic relationships or check a few special cases.

First, we see that a_2 is increasing in ϵ, which is intuitively plausible as the household will want to increase precautionary savings when period-2 income is more risky.

Second, we can inspect the case when $r=0$ so that a_2 simplifies to

$$\begin{aligned}
a_2&=-\frac{2y}{4}+\frac{\sqrt{2^2y^2+8\epsilon^2}}{4}\\
&>-\frac{2y}{4}+\frac{\sqrt{2^2y^2}}{4}=-\frac{2y}{4}+\frac{2y}{4}\\
&=0
\end{aligned}$$

Without uncertainty, we know that $a_2=0$ because $r=0$, $\beta=1$ and $y_1=y_2=y$ so the household has no reason to save. The above expression tells us that in the presence of risk, the household will save a strictly positive amount due to its precautionary savings motive.

Third, we can inspect the corner case of $\epsilon=0$ (with r no longer set to zero). The solution for a_2 then becomes

$$\begin{aligned}
a_2&=-\frac{(2-r)y}{4(1+r)}+\frac{\sqrt{(2+r)^2y^2}}{4(1+r)}\\
&=-\frac{(2-r)y}{4(1+r)}+\frac{(2+r)y}{4(1+r)}=\frac{-2y+ry+2y+ry}{4(1+r)}\\
&=\frac{1}{2}\frac{r}{1+r}y
\end{aligned} \tag{4.29}$$

Recall that in the setting with log preferences, $\beta = 1$ and no uncertainty, in previous units we found that

$$c_1 = \frac{1}{2}\left[y + \frac{y}{1+r}\right] = \frac{1}{2}\frac{2+r}{1+r}y\,,$$

which implies that

$$\begin{aligned} a_2 = y - c_1 = y - \frac{1}{2}\frac{2+r}{1+r}y &= \frac{2(1+r)}{2(1+r)}y - \frac{2+r}{2(1+r)}y = \frac{2y+2ry-2y-ry}{2(1+r)} \\ &= \frac{1}{2}\frac{r}{1+r}y \end{aligned}$$

Reassuringly, this expression is identical to (4.29). ■

We can use numerical methods to go beyond what we were able to do in the previous example without having to impose as many assumptions. To illustrate, one interesting question is how precautionary savings depends on the degree of risk aversion.

Example 4.5 (Precautionary savings and risk aversion). Consider the setting from Example 4.4, except that we no longer impose $\gamma = 1$. How does the household respond to increases in ϵ, and how do these responses depend on a household's risk aversion?

Given how we model risky income y_2 in (4.25), an increase in ϵ represents a mean-preserving spread. Recall from the previous unit that

$$\mathbb{E}y_2 = \frac{1}{2}(y-\epsilon) + \frac{1}{2}(y+\epsilon) = y$$

so $\mathbb{E}y_2$ does not respond to changes in ϵ. On the other hand, the variance is given by

$$\operatorname{Var}(y_2) = \frac{1}{2}[y-\epsilon-y]^2 + \frac{1}{2}[y+\epsilon-y]^2 = \epsilon^2$$

so an increase in ϵ makes income more risky. In a certainty equivalence world, household choices would remain unchanged by a mean-preserving spread, but this is not the case in the precautionary savings model.

Figure 4.6 makes this point. Optimal assets a_2 are increasing in ϵ, and more so for higher levels of risk aversion. Since we assume that $\beta = 1$, $r = 0$ and $y_1 = \mathbb{E}y_2 = y$, the only incentive to save in this model is for precautionary reasons, and these savings increase in a riskier environment.

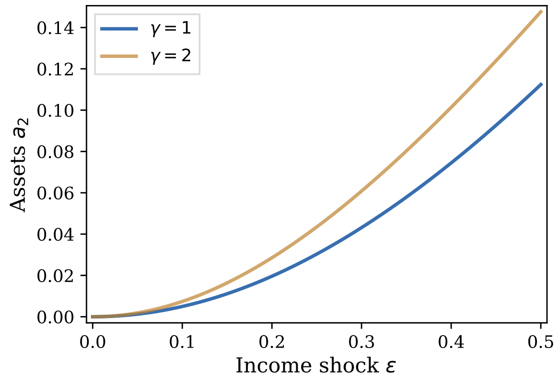


Figure 4.6: Precautionary savings as a function of the RRA coefficient γ and income risk for $\beta = 1, r = 0$.

■

Up until now, our analysis was in partial equilibrium. How would the general equilibrium look in this setting? We illustrate one such equilibrium in the next example.

Example 4.6 (General equilibrium with precautionary savings). Consider an economy populated by an arbitrary number of households who maximise the problem in (4.19) and have income $y_1 = y$ and y_2 given by

$$y_2 = \begin{cases} y - \epsilon & \text{with prob. } \pi \\ y + \epsilon & \text{with prob. } 1 - \pi \end{cases} \tag{4.30}$$

What is the equilibrium interest rate in this economy?

First, note that all households are ex ante identical. Therefore, in equilibrium we cannot have that some households are borrowers while others choose to be savers. We are thus looking for an autarky solution where each household consumes its endowment in each period.

To do this, we proceed in the same way as we did in the first unit. In order to find an interest rate that satisfies each household's optimality condition, we use the Euler equation

$$y^{-\gamma} = \beta(1+r)\mathbb{E}\left[y_2^{-\gamma}\right]$$

evaluated at $c_1 = y$ and $c_2 = y_2$. Using the definition of y_2, the Euler equation can be written as

$$y^{-\gamma} = \beta(1+r)\left[\pi(y-\epsilon)^{-\gamma} + (1-\pi)(y+\epsilon)^{-\gamma}\right] \tag{4.31}$$

The equilibrium gross interest rate is thus given by

$$1 + r = \beta^{-1}\frac{y^{-\gamma}}{\mathbb{E}\left[y_2^{-\gamma}\right]} = \beta^{-1}\frac{y^{-\gamma}}{\pi(y-\epsilon)^{-\gamma} + (1-\pi)(y+\epsilon)^{-\gamma}} \tag{4.32}$$

Even without evaluating this expression, we can get the intuition by looking at Figure 4.4 panel (b) again. Because the marginal utility is a convex function, $\mathbb{E}\left[y_2^{-\gamma}\right]$ will increase as we increase ϵ, and consequently the equilibrium interest rate has to fall. With riskier income, households would want to increase precautionary savings, so a lower interest rate is required to disincentivise them from doing so. This effect is stronger for higher values of the relative risk aversion γ, as illustrated in Figure 4.7.

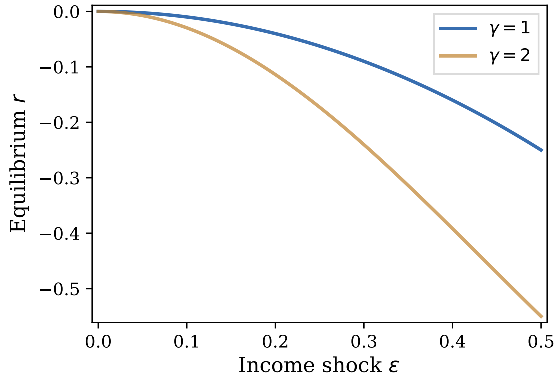


Figure 4.7: Equilibrium interest rate for Example 4.6 as a function of income risk and the RRA coefficient γ for $\beta = 1$, $y = 1$ and $\pi = \frac{1}{2}$.

■

4.5 Main takeaways

In this unit, we studied household choices under uncertainty with incomplete markets in two different settings.

First, in the certainty equivalence model, we concluded that

1. Optimal choices were identical to those without uncertainty if we replaced deterministic values with their expected outcomes.
2. Households did not respond to changes in risk other than changes in the mean. In particular, their optimal choices did not respond to increases in variance.
3. The certainty equivalence model allows for analytical solutions but has many drawbacks that make it less useful for modern macroeconomic modelling.

Second, we studied the precautionary savings model that arises with CRRA preferences and uncertainty. We found that

1. In the presence of risk, households choose to increase their precautionary savings.
2. Optimal solutions no longer coincide with their deterministic counterparts in general.

3. While these models form the backbone of modern macroeconomics, they usually don't allow for analytical solutions and are instead computed numerically.

Appendix 4.1: Detailed solution for Example 4.4

This section contains the detailed steps to solve for optimal savings a_2 in Example 4.4. We begin with the simplified Euler equation from (4.27), repeated here for convenience:

$$\frac{1}{y-a_2} = (1+r)\frac{(1+r)a_2+y}{\left[(1+r)a_2+y\right]^2-\epsilon^2}$$

To solve for a_2 (the only unknown), we bring each denominator to the other side, multiply out all expressions and collection terms to get a standard quadratic equation in a_2:

$$\begin{aligned}\left[(1+r)a_2+y\right]^2-\epsilon^2 &= (y-a_2)(1+r)\left[(1+r)a_2+y\right]\\ (1+r)^2a_2^2+2(1+r)ya_2+y^2-\epsilon^2 &= -(1+r)^2a_2^2+(1+r)^2ya_2+(1+r)y^2-(1+r)ya_2\end{aligned}$$

Collecting terms, we get

$$\begin{aligned}2(1+r)^2a_2^2+(1+r)\left[3-(1+r)\right]ya_2-ry^2-\epsilon^2 &= 0\\ 2(1+r)^2a_2^2+(1+r)(2-r)ya_2-ry^2-\epsilon^2 &= 0\end{aligned}$$

This is a quadratic equation in a_2,

$$Aa_2^2+Ba_2+C=0$$

with

$$\begin{aligned}A &= 2(1+r)^2\\ B &= (1+r)(2-r)y\\ C &= -ry^2-\epsilon^2\end{aligned}$$

which has two solutions,

$$a_2 = -\frac{B}{2A} \pm \frac{\sqrt{B^2-4AC}}{2A}$$

Plugging in these values and simplifying yields

$$\begin{aligned}a_2 &= -\frac{(1+r)(2-r)y}{2\cdot 2(1+r)^2} \pm \frac{\sqrt{\left[(1+r)(2-r)y\right]^2-4\cdot 2(1+r)^2\left(-ry^2-\epsilon^2\right)}}{2\cdot 2(1+r)^2}\\ &= -\frac{(2-r)y}{4(1+r)} \pm \frac{\sqrt{(2-r)^2y^2+8\left(ry^2+\epsilon^2\right)}}{4(1+r)}\end{aligned}$$

Next, we simplify the expression inside the square root:

$$\begin{aligned}(2-r)^2y^2 + 8\left(ry^2+\epsilon^2\right) &= 4y^2 - 4ry^2 + r^2y^2 + 8ry^2 + 8\epsilon^2 \\ &= 4y^2 + 4ry^2 + r^2y^2 + 8\epsilon^2 \\ &= (2+r)^2y^2 + 8\epsilon^2\end{aligned}$$

Finally, the solution for a_2 is

$$a_2 = -\frac{(2-r)y}{4(1+r)} \pm \frac{\sqrt{(2+r)^2y^2+8\epsilon^2}}{4(1+r)}$$

Using economic intuition, we can exclude the solution in which a_2 is always negative irrespective of the interest rate, so we are left with

$$a_2 = -\frac{(2-r)y}{4(1+r)} + \frac{\sqrt{(2+r)^2y^2+8\epsilon^2}}{4(1+r)}$$

Appendix 4.2: Parametrisation of quadratic preferences

In this section, we illustrate how the parameters governing a generic quadratic function map into the parameters of the quadratic utility function in (4.8). A generic quadratic function has three parameters, denoted here by A, B and C, where the functional form is given by

$$f(x) = A(x-B)^2 + C \tag{4.33}$$

In this case, A controls the slope of the function, while B shifts its extremum (maximum or minimum) horizontally, and C shifts the function vertically. The extremum where the function is maximised (for negative A) or minimised (for positive A) is given by $(x^*, y^*) = (B, C)$. This is illustrated in Figure 4.8.

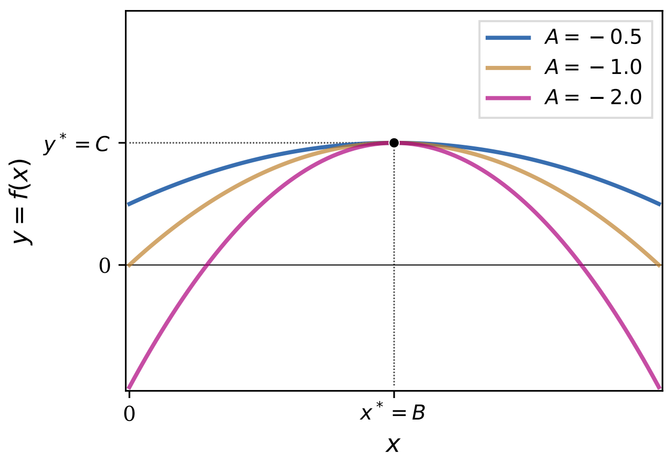


Figure 4.8: Quadratic utility functions for various slope parameters for $B = C = 1$.

Expanding the quadratic term in (4.33), we see that

$$\begin{aligned} f(x) &= A(x-B)^2 + C \\ &= A(x^2 - 2xB + B^2) + C \\ &= Ax^2 - 2ABx + AB^2 + C \end{aligned}$$

If we compare this to our quadratic utility function in (4.8), we find that

$$\begin{aligned} \alpha &= -2AB \\ -\frac{\delta}{2} &= A \\ 0 &= AB^2 + C \end{aligned}$$

We see that $\delta = -2A$ maps directly into the parameter governing the slope, while $\alpha = -2AB$ influences both the slope and the horizontal location of the utility function. Moreover, it's now easy to see that $\alpha/\delta = -2AB/(-2A) = B$ corresponds to the point where utility attains its global maximum. Note that we omit the constant $AB^2 + C$ from the quadratic utility function since it has no effect on optimal choices.

Exercises

Exercise 4.1 (Interest rate and income risk). Consider the following household problem with log preferences and income risk:

$$\max_{c_1, c_2, a_2} \quad \log(c_1) + \beta \mathbb{E}\left[\log(c_2)\right] \tag{E.1}$$

$$\text{s.t.} \quad c_1 + a_2 = y \tag{E.2}$$

$$c_2 = (1+r)a_2 + y_2 \tag{E.3}$$

where period-2 income is given by

$$y_2 = \begin{cases} y - \epsilon & \text{with prob. } \frac{1}{2} \\ y + \epsilon & \text{with prob. } \frac{1}{2} \end{cases} \tag{E.4}$$

with $0 < \epsilon < y$. You can ignore any borrowing constraints for this exercise.

(a) Derive the Euler equation for this problem.

(b) Assume that the economy is populated by an arbitrary number of ex ante identical households. Derive the equilibrium interest rate as a function of the variance of period-2 income, $\text{Var}\left(y_2\right)$.

Note: With this parametrisation, it is possible to obtain an intuitive expression for r.

(c) Let $\beta = 1$ and $y = 1$. Plot the equilibrium interest rate on the y-axis against the income variance $\text{Var}\left(y_2\right)$ on the x-axis. Provide an intuition for the slope of this relationship!

Exercise 4.2 (General equilibrium with ex ante heterogeneity). Consider an economy with two households, A and B, who solve

$$\max_{c_1, c_2, a_2} \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left[\frac{c_2^{1-\gamma}}{1-\gamma}\right]$$

$$\text{s.t.} \quad c_1 + a_2 = y$$

$$c_2 = (1+r)a_2 + y_2$$

$$a_2 \geq 0 \tag{E.5}$$

where we impose the no-borrowing constraint (E.5). Period-2 income for household A is given by

$$y_2^A = \begin{cases} y - \epsilon & \text{with prob. } \frac{1}{2} \\ y + \epsilon & \text{with prob. } \frac{1}{2} \end{cases}$$

with $0 < \epsilon < \frac{y}{2}$, whereas for household B it's

$$y_2^B = \begin{cases} y - 2\epsilon & \text{with prob. } \frac{1}{2} \\ y + 2\epsilon & \text{with prob. } \frac{1}{2} \end{cases}$$

(a) Assume that we are looking for an autarky equilibrium in which each household consumes its endowment.

Derive the equation that pins down the equilibrium interest rate!

(b) Let $\beta = 1$, $\gamma = 2$, $y = 1$ and $\epsilon = 0.1$. Compute the value for the equilibrium interest rate!

(c) Now assume that period-2 income for A is given by

$$y_2^A = \begin{cases} y - 1.5\epsilon & \text{with prob. } \frac{1}{2} \\ y + 1.5\epsilon & \text{with prob. } \frac{1}{2} \end{cases}$$

Using the same parametrisation as above, how does the autarky equilibrium interest rate change?

Exercise 4.3 (Mean-preserving spread in general equilibrium). Consider the consumption-savings problem under uncertainty,

$$\begin{aligned} &\max_{c_1, c_2, a_2} \quad u(c_1) + \beta \mathbb{E} u(c_2) \\ \text{s.t.} \quad & c_1 + a_2 = y \\ & c_2 = (1 + r)a_2 + y_2 \end{aligned}$$

where y_2 is stochastic such that

$$y_2 = \begin{cases} y - \epsilon & \text{with prob. } \frac{1}{2} \\ y + \epsilon & \text{with prob. } \frac{1}{2} \end{cases}$$

and $0 < \epsilon < \frac{y}{2}$. You can ignore any borrowing constraints in this exercise. The Euler equation for this problem is the usual

$$u'(c_1) = \beta(1 + r)\mathbb{E}u'(c_2) \ .$$

We want to solve for general equilibrium with autarky, i.e., each household consumes its endowment each period.

(a) Write down the Euler equation used to pin down the equilibrium interest rate for the scenario in which the household has

(1) quadratic utility

(2) CRRA utility

(b) Illustrate the effect of a mean-preserving spread if period-2 income is instead given by

$$y_2 = \begin{cases} y - 2\epsilon & \text{with prob. } \frac{1}{2} \\ y + 2\epsilon & \text{with prob. } \frac{1}{2} \end{cases} \tag{E.6}$$

using graphs that are similar to Figure 4.4, one for the quadratic and one for the CRRA case. On the x-axis, indicate the points y, $y \pm \epsilon$ and $y \pm 2\epsilon$, and plot the marginal utility and its expected value.

(c) What happens to the expectation in the Euler equation in the quadratic and CRRA cases? How do optimal savings respond? How does the equilibrium interest rate respond?

Provide a qualitative answer and the intuition for the underlying mechanism!

Exercise 4.4 (Equilibrium with government transfers). Consider an economy with two households, A and B, who solve

$$\begin{aligned} \max_{c_1, c_2, a_2} \quad & \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left[\frac{c_2^{1-\gamma}}{1-\gamma}\right] \\ \text{s.t.} \quad & c_1 + a_2 = y \\ & c_2 = (1+r)a_2 + y_2 \end{aligned}$$

Moreover, assume that these households have perfectly negatively correlated income as shown in Table E.1, where $0 < \epsilon < y$ and each outcome has a probability of $\pi = \frac{1}{2}$.

Household	Income in $t = 1$	Income in $t = 2$	
		State b (prob. $\frac{1}{2}$)	State g (prob. $\frac{1}{2}$)
A	y	$y - \epsilon$	$y + \epsilon$
B	y	$y + \epsilon$	$y - \epsilon$

Table E.1: Income in economy for Exercise 4.4.

(a) State the Euler equation characterising the solution to this problem. Explain why the Euler equation is the same for A and B.

(b) Derive the equilibrium condition for the interest rate in this economy.

(c) Let $\beta = 1$, $\gamma = 2$, $y = 1$ and $\epsilon = 0.5$. Compute the equilibrium interest rate for this economy.

(d) Assume now that a redistributive government levies a lump-sum tax of $\frac{\epsilon}{2}$ on the lucky household (which can be either A or B) and transfers these funds to the unlucky one.

How does this policy change equilibrium savings? How is the equilibrium interest rate affected?

Exercise 4.5 (Complete vs. incomplete markets). Consider an economy with two households, A and B, who have log preferences and face income uncertainty in the second period:

$$\begin{aligned} \max_{c_1, c_2, a_2} \quad & \log(c_1) + \beta \mathbb{E}\left[\log(c_2)\right] \\ \text{s.t.} \quad & c_1 + a_2 = y \\ & c_2 = (1+r)a_2 + y_2 \end{aligned}$$

where y_2 is stochastic as shown in Table E.2 with $0 < \epsilon < \frac{y}{2}$. You can disregard any borrowing constraints.

Both households are ex ante identical, but ex post their income realisations in period 2 are perfectly negatively correlated so that in the aggregate, income is constant at Y.

Household	Income in $t = 1$	Income in $t = 2$	
		State b (prob. $\frac{1}{2}$)	State g (prob. $\frac{1}{2}$)
A	y	$y - \epsilon$	$y + \epsilon$
B	y	$y + \epsilon$	$y - \epsilon$
Aggregate	Y	Y	Y

Table E.2: Income in economy for Exercise 4.5.

(a) Write down an equation with a single unknown, r, that allows you to determine the equilibrium interest rate (you don't need to explicitly solve for r).

(b) Can you say whether $\beta(1 + r) > 1$, $\beta(1 + r) = 1$ or $\beta(1 + r) < 1$? What is the intuition behind your finding?

(c) Assume that the households are allowed to trade in Arrow bonds contingent on the states b and g with prices q_b and q_g, respectively, i.e., there are complete markets like in unit 3. Each household now solves

$$\max_{c_1, c_{2b}, c_{2g}} \quad \log(c_1) + \beta \mathbb{E}\left[\log(c_2)\right]$$
$$\text{s.t.} \quad c_1 + q_b c_{2b} + q_g c_{2g} = y + q_b y_{2b} + q_g y_{2g}$$

where

$$y_{2b} = \begin{cases} y - \epsilon & \text{for household } A \\ y + \epsilon & \text{for household } B \end{cases}$$

and

$$y_{2g} = \begin{cases} y + \epsilon & \text{for household } A \\ y - \epsilon & \text{for household } B \end{cases}$$

What are the equilibrium allocations $(c_1^i, c_{2b}^i, c_{2g}^i)$ for each household $i = A, B$? How do these compare to the incomplete markets equilibrium discussed above?

Hint: The complete markets part of this exercise is almost identical to Example 6 of unit 3.

(d) Solve for the risk-free interest rate in the complete markets economy. How does it compare to the equilibrium interest rate in the incomplete markets setting?

(e) Consider a mean-preserving spread for period-2 income such that income now fluctuates with $\pm 2\epsilon$ instead of $\pm\epsilon$.

How do the equilibrium allocations and prices in the complete markets economy respond? Provide an economic intuition for your answer!

5 Overlapping generations models

5.1 Introduction

In unit 2, we studied life cycle models in which agents live for two or more periods. However, our analysis was purely in partial equilibrium. In order to take such models to general equilibrium, we need to impose an overlapping generations (OLG) structure in which at any point in time, multiple cohorts of different ages are alive in the economy.

In the OLG models discussed in this unit, we will restrict ourselves to agents who live for only two or three periods. In the simplest case of two periods, there are two cohorts which we call *young* (1^{st} period of life denoted with subscript 1) and the *old* (2^{nd} period of life denoted with subscript 2). This situation is depicted in Figure 5.1. At any time t

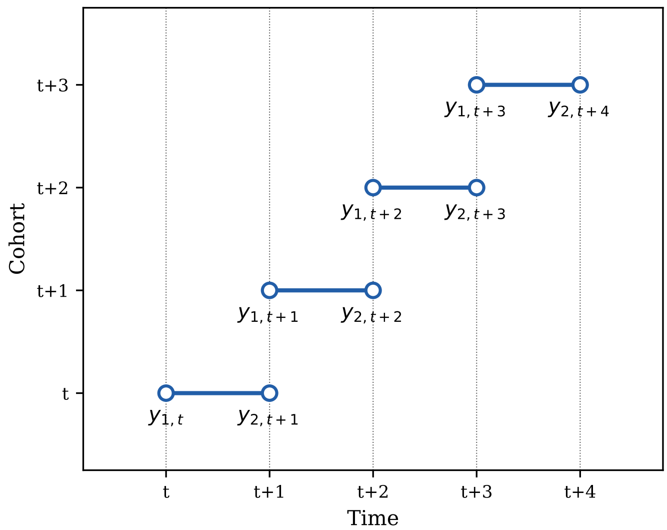


Figure 5.1: Cohort structure in OLG model with agents who live for two periods. (y_1, y_2) denotes endowments agents receive when young and old, respectively.

exactly two cohorts are alive in the economy: the old (cohort born in $t-1$) who receive the endowment $y_{2,t}$ and the young (cohort born in t) who receive $y_{1,t}$.

In what follows, we impose that there is only one household in each cohort. We could call this setup a "representative cohort" model because in principle we allow for arbitrarily many *identical* households in each cohort as long as the cohorts are of equal size.

We assume that the economy itself exists indefinitely and is stationary, i.e., all aggregate variables and the distribution of households are invariant over time. We perform our analysis based on a snapshot at some arbitrary t and will omit the subscript t in the remainder of this unit.

5.2 Endowment economies without a government

5.2.1 Economy with two overlapping cohorts

So far, we haven't made any assumptions about the endowment. Let's first examine the case when $y_2 = 0$ and the household maximizes

$$\begin{aligned} \max_{c_1, c_2, a_2} \quad & u(c_1) + \beta u(c_2) \\ \text{s.t.} \quad & c_1 + a_2 = y_1 \\ & c_2 = (1+r)a_2 \end{aligned} \tag{5.1}$$

This is a well-defined problem in partial equilibrium, but does not work in general equilibrium in a pure endowment economy. To see this, we first conclude that the young household is forced to choose positive savings $a_2 > 0$, as otherwise it will starve in the second period — there is no other income in (5.1). Because there are no assets in positive net supply, any savings must be matched by borrowing of the other household in the economy. However, the only other household alive is the old household, which cannot borrow because it won't be alive next period to repay that debt. Consequently, the young household will be unable to save and studying this economy is not particularly insightful.

Of course, this is an artefact of assuming that households live for only two periods which we relax in the next section. In a full-fledged model with 60 periods similar to those we studied in unit 2, young households would want to borrow while middle-aged ones would want to save for retirement. We would consequently observe inter-cohort trade even in a pure endowment economy.

However, complex OLG models with realistic demographics are beyond the scope of this course. We will therefore examine other ways for consumption smoothing in simplified settings with only two cohorts:

1. The most obvious one is to give households positive endowments when old, e.g., due to some (unmodelled) social security system.

2. We can introduce a government which facilitates intergenerational transfers, a topic we cover in the next section.

3. Lastly, we can introduce capital which is an asset in positive net supply and allows households to save even if no other agent in the economy wants to borrow. Even though this is the most plausible assumption, it also adds complexity, so we don't cover this type of model in this unit.

In the remainder of this section, we examine the autarky equilibrium that arises if households receive positive endowments in both periods.

Example 5.1 (Endowments in both periods). Consider a household which lives for two periods and solves the following problem:

$$\begin{aligned} \max_{c_1, c_2, a_2} \quad & u(c_1) + \beta u(c_2) \\ \text{s.t.} \quad & c_1 + a_2 = y_1 \\ & c_2 = (1+r)a_2 + y_2 \end{aligned}$$

where $y_1 > 0$ and $y_2 > 0$ are income received when young and old, respectively.

We have solved this problem multiple times, so we immediately conclude that the Euler equation is given by

$$u'(c_1) = \beta(1+r)u'(c_2)$$

As discussed above, the old household can neither borrow (because it would not be able to repay its debt), nor does it want to save since it will not be around to consume these savings later. Consequently, the young household will not be able to save or borrow in equilibrium either. The only equilibrium we are able to find in this setting is therefore the autarky equilibrium with $c_1 = y_1$ and $c_2 = y_2$. The equilibrium interest rate r is thus pinned down by

$$r = \frac{u'(y_1)}{\beta u'(y_2)} - 1 \,.$$

Imposing CRRA preferences, this expression can be written as

$$r = \frac{1}{\beta} \frac{y_1^{-\gamma}}{y_2^{-\gamma}} - 1 = \frac{1}{\beta} \left[\frac{y_2}{y_1}\right]^{\gamma} - 1 \,.$$

■

The above example does not introduce anything novel compared to the two-period consumption-savings problem we covered in the previous units, so let's next introduce a third period of life.

5.2.2 Economy with three overlapping cohorts

Extending the environment from the previous section to agents who life for three periods opens up the possibility of equilibria with trade. In each period t, the economy is now populated by three generations, and we can interpret each period as representing 20–25 years. This environment is depicted in Figure 5.2.

Household problem (partial equilibrium). If we ignore general equilibrium for the moment, then the household problem in this setting is exactly the same as discussed in unit 2. For completeness, let's work through the solution again, assuming that the

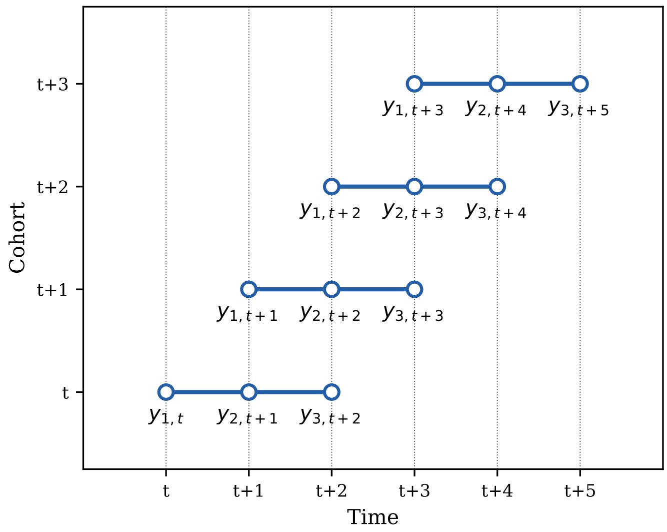


Figure 5.2: Cohort structure in OLG model with agents who live for *three* periods and receive endowments (y_1, y_2, y_3).

household solves

$$
\begin{aligned}
\max_{c_1, c_2, c_3, a_2, a_3} \quad & u(c_1) + \beta u(c_2) + \beta^2 u(c_3) \\
\text{s.t.} \quad c_1 + a_2 &= y_1 \\
c_2 + a_3 &= (1+r)a_2 + y_2 \\
c_3 &= (1+r)a_3 + y_3
\end{aligned}
$$

As before, in the terminal period 3, the household is not allowed to borrow because it won't be around to repay its debt, and it has no incentive to save. However, we can find equilibria where households of ages 1 and 2 engage in lending/saving and borrowing because the age-2 household wants to save for retirement in period 3, while a household of age 1 (young worker) faces an upward-sloping income trajectory and wants to borrow against future income. We illustrate such a scenario below.

The consolidated lifetime budget constraint is given by

$$
c_1 + \frac{c_2}{(1+r)} + \frac{c_3}{(1+r)^2} = y_1 + \frac{y_1}{(1+r)} + \frac{y_3}{(1+r)^2}
$$

so that we can eliminate savings a_2 and a_3 and state the Lagrangian only in terms of consumption,

$$
\mathcal{L} = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \lambda \left[y_1 + \frac{y_1}{(1+r)} + \frac{y_3}{(1+r)^2} - c_1 - \frac{c_2}{(1+r)} - \frac{c_3}{(1+r)^2} \right]
$$

This gives rise to the usual first-order conditions,

$$\frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta u'(c_2) - \frac{\lambda}{1+r} = 0$$
$$\frac{\partial \mathcal{L}}{\partial c_3} = \beta^2 u'(c_2) - \frac{\lambda}{(1+r)^2} = 0$$

which we can combine to get *two* Euler equations which characterise the intertemporal consumption choice between ages 1 and 2, as well as ages 2 and 3:

$$u'(c_1) = \beta(1+r)u'(c_2)$$
$$u'(c_2) = \beta(1+r)u'(c_3)$$

With CRRA preferences, these Euler equations become,

$$c_1^{-\gamma} = \beta(1+r)c_2^{-\gamma}$$
$$c_2^{-\gamma} = \beta(1+r)c_3^{-\gamma}$$

We can then express c_2 and c_3 as a function of c_1,

$$c_2 = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} c_1$$
$$c_3 = \left[\beta(1+r)\right]^{\frac{1}{\gamma}} c_2 = \left[\beta(1+r)\right]^{\frac{2}{\gamma}} c_1$$

These can then be substituted into the lifetime budget constraint to solve for c_1 as a function of exogenous income, parameters and the equilibrium interest rate that remains to be determined. For example, for log preferences with $\gamma = 1$, optimal consumption is

$$c_1 = \frac{1}{1+\beta+\beta^2}\left[y_1 + \frac{y_2}{(1+r)} + \frac{y_3}{(1+r)^2}\right] \tag{5.2}$$
$$c_2 = \frac{\beta(1+r)}{1+\beta+\beta^2}\left[y_1 + \frac{y_2}{(1+r)} + \frac{y_3}{(1+r)^2}\right] \tag{5.3}$$
$$c_3 = \frac{\beta^2(1+r)^2}{1+\beta+\beta^2}\left[y_1 + \frac{y_2}{(1+r)} + \frac{y_3}{(1+r)^2}\right] \tag{5.4}$$

General equilibrium. In order to find the equilibrium interest rate r we need to impose market clearing in the savings market. Because each cohort consists of a single representative household, this requires that the amount a household wishes to borrow (or lend) at age 1 has to be identical to the amount a household chooses to lend (or borrow) at age 2 so that in the aggregate, savings are exactly zero.[1] Therefore, in equilibrium we require that

$$-a_2 = a_3$$

[1] Recall that it a model without physical capital or government bonds, assets have to be in zero net supply in the aggregate.

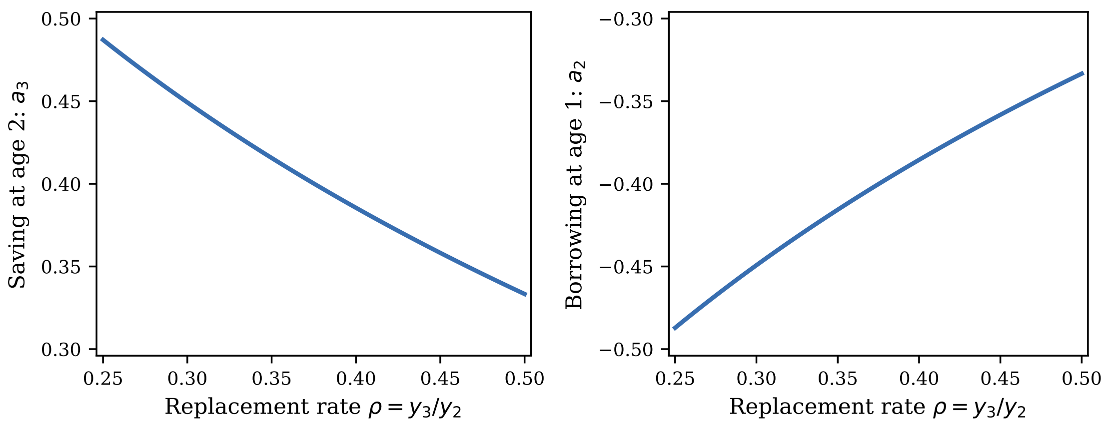
or, plugging in the expressions for the budget constraints at ages 1 and 3,

$$-(y_1 - c_1) = \frac{1}{1+r}(y_3 - c_3)$$

Once we plug in the optimal c_1 and c_3 from (5.2) and (5.4), this becomes a nonlinear equation in r that has no obvious analytical solution. We therefore need to resort to numerical methods to find the equilibrium interest rate, which we do next.

Example 5.2 (General equilibrium with three cohorts). Continuing with the three-cohort setup from above, let $\beta = 1$, $\gamma = 1$ and assume that the household faces an upward-sloping income profile at ages 1 and 2 (which represent working age) with $y_1 = 1$, $y_2 = 2$. We impose that in the terminal (retirement) period, the household receives a fraction $\rho = y_3/y_2$ of its age-2 income where $\rho < 1$. The household therefore has an incentive to save for retirement, whereas it has a incentive to borrow at age 1 as it faces an upward-sloping income profile.

The optimal borrowing/savings levels for this economy are shown in Figure 5.3, and the corresponding equilibrium interest rate is plotted in Figure 5.4. We see that the household wants to save more for retirement if the replacement rate ρ is low, say 25% of age-2 income. Correspondingly, in equilibrium a young household has to be willing to borrow a higher amount to clear the savings market, so the equilibrium interest rate needs to decrease (it is in fact negative) to support this level of borrowing.



(a) Saving at age 2: a_3

(b) Borrowing at age 1: a_2

Figure 5.3: Borrowing/saving plotted against the replacement rate $\rho = y_3/y_2$ of retirement income.

■

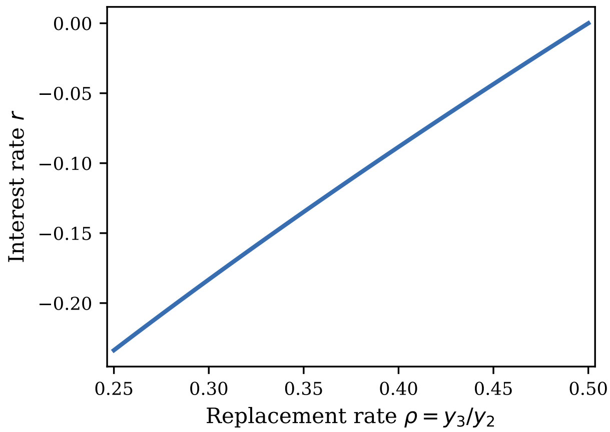


Figure 5.4: Equilibrium interest rate plotted against the replacement rate $\rho = y_3/y_2$ of retirement income.

5.3 OLG with a government

5.3.1 Government debt

In the previous section, we saw that if agents lived for only two periods, the only possible equilibrium was one without trade. One way to allow for saving is to provide an asset in positive net supply so that households can save even if no one in the economy wants to borrow. Such an asset could be physical capital if we are willing to add production. Alternatively, in a pure endowment economy we can introduce government debt which households can hold in order to save for future periods. The infinitely-lived government thus enables households with (short) finite lives to smooth consumption.

Assume that the government has a stock of debt b_t on which it pays an interest rate of r. Additionally, the government taxes households to raise revenue which can be used to pay interest or pay down debt. Its dynamic budget constraint is therefore given by

$$\underbrace{b_{t+1} + \tau}_{\text{Revenues}} = \underbrace{(1+r)b_t}_{\text{Debt repayment}}$$

where τ is the tax revenue and b_{t+1} is newly issued debt in period t. Assuming that the debt level is constant, the government budget simplifies to

$$b + \tau = (1+r)b \quad \Longrightarrow \quad \tau = rb \tag{5.5}$$

The government thus rolls over its stock of debt and needs to raise just enough revenue to service the interest payments rb. In the next example, we examine how government debt can help households transfer resources between their two periods of life.

Example 5.3 (Endowment economy with government debt). Assume that households solve the following two-period problem:

$$\begin{aligned} \max_{c_1, c_2, a_2} \quad & \log(c_1) + \beta \log(c_2) \\ \text{s.t.} \quad & c_1 + a_2 = y_1 - \tau \\ & c_2 = (1+r)a_2 \end{aligned}$$

The household can choose to save an amount a_2 in government debt and has to pay income taxes τ when young.[2] As usual, the household takes the interest rate r and the income tax τ as given, and these will be determined in general equilibrium.

Household problem (partial equilibrium). This is a standard consumption-savings problem with the lifetime budget constraint given by

$$c_1 + \frac{c_2}{1+r} = y_1 - \tau \ .$$

The Euler equation governing the optimal intertemporal consumption decision is

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2} \ . \tag{5.6}$$

Solving for $c_2 = \beta(1+r)c_1$ and plugging into the lifetime budget constraint, we find that

$$\begin{aligned} c_1 + \frac{\beta(1+r)c_1}{1+r} &= y_1 - \tau \\ \implies \quad c_1 &= \frac{1}{1+\beta}\left[y_1 - \tau\right] \end{aligned} \tag{5.7}$$

Consequently, optimal bond savings are given by

$$a_2 = y_1 - \tau - c_1 = y_1 - \tau - \frac{1}{1+\beta}\left[y_1 - \tau\right] = \frac{\beta}{1+\beta}\left[y_1 - \tau\right] \ . \tag{5.8}$$

General equilibrium. We take the stock of debt b as the government's policy variable and determine the remaining two unknowns, τ and r, such that the government budget balance and the households' optimality conditions are satisfied — after all, the households must be willing to hold this amount of debt at the equilibrium interest rate given their disposable income $y_1 - \tau$. Note that we could have alternatively taken τ as the policy variable and determined b and r in equilibrium, arriving at the same solution.

Since we assume there is a single (representative) young household, bond market clearing requires that

$$a_2 = b \ .$$

[2] Income taxes are usually proportional to income, i.e., the tax is given by $\tau \cdot y_1$. Since the endowment is exogenous in this example, it makes no difference whether the tax is modelled as lump-sum or a fraction of y_1.

There are several ways to find r and τ from here. For example, imposing bond market clearing in in (5.8), we get

$$b = \frac{\beta}{1+\beta}\left[y_1 - \tau\right] .$$

Using the government budget constraint (5.5) to substitute for τ we get an equation in one unknown,

$$b = \frac{\beta}{1+\beta}\left[y_1 - rb\right]$$

which we can solve for r:

$$\begin{aligned}
\frac{1+\beta}{\beta} b &= y_1 - rb \\
\frac{1+\beta}{\beta} &= \frac{y_1}{b} - r \\
\implies \quad r &= \frac{y_1}{b} - \frac{1+\beta}{\beta}
\end{aligned} \tag{5.9}$$

Finally, plugging r back into the government budget constraint (5.5) yields the equilibrium tax rate,

$$\tau = rb = y_1 - \frac{1+\beta}{\beta} b . \tag{5.10}$$

Note that we could have alternatively found the equilibrium interest from the households Euler equation (5.6) after plugging in the household and government budget constraints as well as bond market clearing:

$$\begin{aligned}
\frac{1}{y_1 - b - rb} &= \beta(1+r)\frac{1}{(1+r)b} \\
\frac{1}{y_1 - (1+r)b} &= \beta\frac{1}{b} \\
y_1 - (1+r)b &= \frac{b}{\beta} \\
\frac{y_1}{b} - (1+r) &= \frac{1}{\beta} \\
\implies \quad r &= \frac{y_1}{b} - \frac{1+\beta}{\beta}
\end{aligned}$$

The optimal consumption while young can be obtained by plugging the income tax (5.10) into (5.7):

$$c_1 = \frac{1}{1+\beta}\left[y_1 - \tau\right] = \frac{1}{1+\beta}\left[y_1 - \left(y_1 - \frac{1+\beta}{\beta}b\right)\right] = \frac{1}{\beta}b \tag{5.11}$$

Lastly, we find the optimal consumption in old age by substituting r from (5.9) into the budget constraint:

$$c_2 = (1+r)b = \left[1 + \frac{y_1}{b} - \frac{1+\beta}{\beta}\right] b = y_1 - \frac{1}{\beta}b \tag{5.12}$$

These consumption choices are depicted in Figure 5.5 as a function of the debt level which we express relative to income, b/y_1. The corresponding equilibrium income tax τ is shown in panel (a) of Figure 5.6, while the equilibrium interest rate is depicted in panel (b).

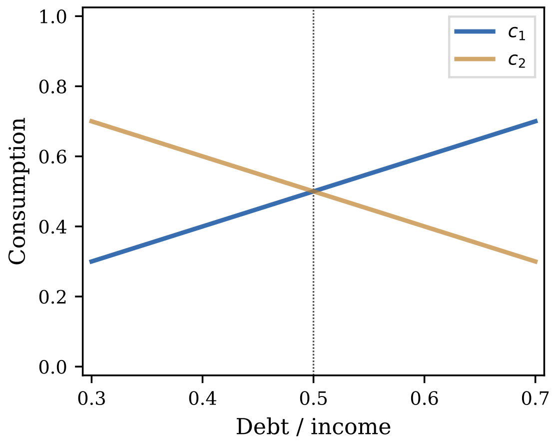
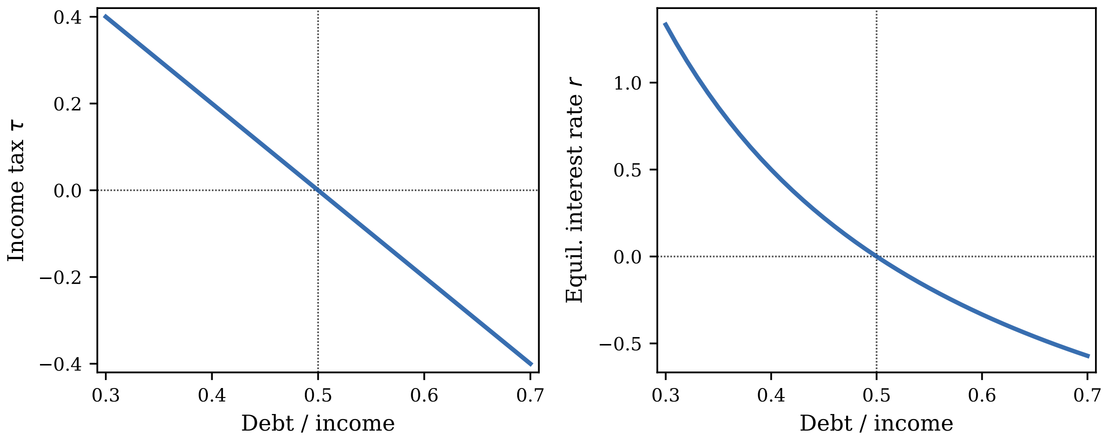


Figure 5.5: Consumption plotted against the debt-to-income ratio b/y_1 for $\beta = 1$ and $y_1 = 1$.



(a) Income tax τ

(b) Interest rate r

Figure 5.6: Tax level and equilibrium interest rate plotted against the debt-to-income ratio b/y_1 for $\beta = 1$ and $y_1 = 1$.

The finding that the interest rate is extremely high for low levels of debt seems puzzling. At least in partial equilibrium, one would conjecture that if the government maintains a low level of debt, and thus young households should choose to save little, the required interest rate needs to be very low. However, another way to bring down savings to be in line with low government debt is to tax away the young household's endowment, which is what happens in this equilibrium.

To illustrate, let $\beta = 1$, $y_1 = 1$, and assume that the government wants to impose a debt/income level of 0.3 which corresponds to the left-most point in the preceding figures. To do this, the government sets an income tax of $\tau = 0.4$. Moreover, the prevailing equilibrium interest rate is given by $r = 1.33$. The household is thus left with a disposable income of $y_1 - \tau = 0.6$ and saves half of it as prescribed by (5.8). Next period, it consumes its gross asset return which is given by $c_2 = (1+r)a_2 = 2.33 \times 0.3 = 0.7$. The government has to pay $r\,a_2 = 0.4$ in interest payments which are exactly offset by the income tax it collects to balance the government budget. ■

So far, we treated the government debt level b as exogenous. In reality, this is a policy variable that the government can control, so the question arises which debt level the government should choose.

Example 5.4 (Optimal level of government debt). Consider the economy in Example 5.3, where we took the level of government debt b as given and solved for the equilibrium allocation and interest rate. Figure 5.5 illustrates that households make vastly different consumption choices depending on the debt-to-income ratio, so a natural follow-up question pertains to the optimal level of debt.

Assuming that the government values the welfare of all cohorts equally, we can pin down the welfare-maximising level of debt by maximising the utility of any single cohort. The government thus solves

$$\max_{b \in [0, \beta y_1]} \quad \log(c_1^*) + \beta \log(c_2^*)$$

where c_1^* and c_2^* are the *optimal* households choices given b, which themselves are functions of b given by (5.11) and (5.12),

$$c_1^* = \frac{1}{\beta} b$$
$$c_2^* = y_1 - \frac{1}{\beta} b$$

The government thus perfectly anticipates that households adjust their choices optimally as it varies the level of debt. Note that we also impose the constraint that $b \in [0, \beta y_1]$. This ensures that neither c_1 nor c_2 are negative (which would not be optimal anyway).

Plugging in the optimal consumption choices, the government maximises

$$\max_{b \in [0, \beta y_1]} \quad \log\left(\beta^{-1} b\right) + \beta \log\left(y_1 - \beta^{-1} b\right) \; .$$

Figure 5.7 illustrates this maximisation problem for $\beta = y_1 = 1$.

Note that we can get rid of the additive constant that comes from $\log\left(\beta^{-1} b\right) = \log(b) - \log(\beta)$, so the problem can equivalently be stated as

$$\max_{b \in [0, \beta y_1]} \quad \log(b) + \beta \log\left(y_1 - \beta^{-1} b\right) \; .$$

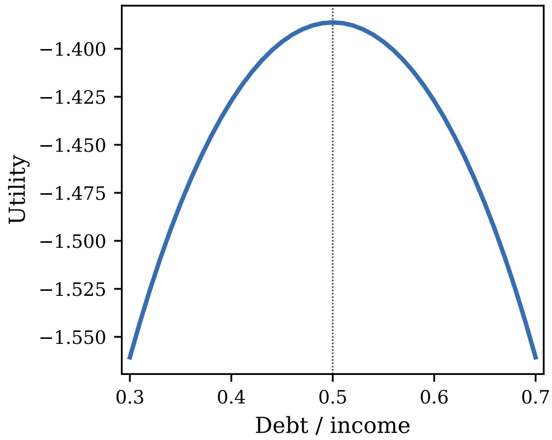


Figure 5.7: Household utility as a function of debt-to-income ratio for $\beta = 1$ and $y_1 = 1$.

We set up the Lagrangian in the usual way, taking into account the two constraints $b \geq 0$ and $b \leq \beta y_1$ with Lagrange multipliers λ_1 and λ_2, respectively:

$$\mathcal{L} = \log(b) + \beta \log\left(y_1 - \beta^{-1} b\right) + \lambda_1 b + \lambda_2 (\beta y_1 - b)$$

For an interior solution, the first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial b} = \frac{1}{b} - \beta \frac{\beta^{-1}}{y_1 - \beta^{-1} b} = 0$$

since we know that at the optimum the Lagrange multipliers are $\lambda_1 = \lambda_2 = 0$. Solving the above equation for b is straightforward:

$$\begin{aligned}
\frac{1}{b} &= \frac{1}{y_1 - \beta^{-1} b} \\
b &= y_1 - \frac{1}{\beta} b \\
\frac{1+\beta}{\beta} b &= y_1 \\
\implies \quad b^* &= \frac{\beta}{1+\beta} y_1
\end{aligned}$$

where we use a star to denote the *welfare-maximising* debt level, b^*. Recalling the optimal expression for savings in a two-period model without borrowing constraints, it may not come as a surprise that the optimal debt level exactly replicates this savings level. By the same logic, optimal consumption at the optimal debt level is

$$\begin{aligned}
c_1 &= \frac{1}{\beta} b^* = \frac{1}{1+\beta} y_1 \\
c_2 &= y_1 - \frac{1}{\beta} b^* = \frac{\beta}{1+\beta} y_1
\end{aligned}$$

Note that c_1 again coincides with what the household would have optimally chosen if it was free to save.

Finally, the equilibrium interest rate is obtained by plugging b^* into (5.9),

$$r = \frac{y_1}{b^*} - \frac{1+\beta}{\beta} = \frac{y_1}{\frac{\beta}{1+\beta} y_1} - \frac{1+\beta}{\beta} = 0$$

and the income tax τ follows from the government budget constraint, $\tau = r \cdot b^* = 0$.

The equilibrium at the optimal debt level is indicated by dotted lines in Figure 5.5 and Figure 5.6 for the parameters $\beta = y_1 = 1$. Because the household does not discount future utility, the optimal consumption allocation in this case prescribes that the household consumes exactly half of its endowment y_1 in each period.

■

5.3.2 Pension system with exogenous labour supply

Instead of issuing bonds, the government can also use the tax and transfer system to redistribute resources across cohorts. The prime example of such redistribution is the pension system, which we discuss in the next example.

Example 5.5 (Endowment economy with PAYGO pension system). Consider a household which lives for two periods and solves

$$\begin{aligned}
&\max_{c_1, c_2, a_2} \quad u(c_1) + \beta u(c_2) \\
\text{s.t.} \quad & c_1 + a_2 = y_1 - \tau \\
& c_2 = (1+r) a_2 + \tau
\end{aligned}$$

where $0 < \tau < y_1$ is a tax levied on the young cohort which we can interpret as a payroll tax on income earned while young.[3]

Compared to Example 5.1, nothing changes in terms of savings since there is no asset in positive net supply. The old still cannot borrow or save, and hence neither can the young. However, now consumption in old age is financed by the transfer τ. We can interpret this as old-age consumption being financed by a "pay as you go" (PAYGO) retirement system. Because we assume that all cohorts are of equal size, the government budget balance requires that

$$\underbrace{\tau}_{\text{Payroll tax revenues}} = \underbrace{\tau}_{\text{Pensions}}$$

which is automatically satisfied.

[3] Payroll taxes are often proportional to income, i.e., the tax is given by $\tau \cdot y_1$. Since the endowment is exogenous in this example, it makes no difference whether the tax is modelled as lump-sum or a fraction of y_1.

In equilibrium, we must have $c_1 = y_1 - \tau$ and $c_2 = \tau$. The interest rate is therefore given by

$$r = \frac{u'(y_1 - \tau)}{\beta u'(\tau)} - 1$$

where we substituted into the Euler equation and solved for r. ■

By varying τ, the government can directly control household consumption. A natural follow-up question concerns the welfare-maximising level of payroll taxes which is a policy variable set by the government and taken as given by households.

Example 5.6 (Optimal payroll tax with PAYGO pension system). Continuing with the economy from Example 5.5, we now want to determine the optimal level of τ. Assuming that the government wants to maximise welfare for each cohort, it solves

$$\max_{\tau \in [0, y_1]} \quad u(y_1 - \tau) + \beta u(\tau)$$

which gives rise to the Lagrangian

$$\mathcal{L} = u(y_1 - \tau) + \beta u(\tau) + \lambda_1 \tau + \lambda_2 (y_1 - \tau)$$

with Lagrange multipliers λ_1 and λ_2 for the constraints $\tau \geq 0$ and $\tau \leq y_1$, respectively. As before, we know that the government will choose an interior payroll tax $0 < \tau < y_1$ so as to ensure positive consumption of the young and old, so we can ignore these constraints. The first-order condition is given by

$$\frac{\partial \mathcal{L}}{\partial \tau} = -u'(y_1 - \tau) + \beta u'(\tau) = 0$$

or, assuming CRRA preferences,

$$-(y_1 - \tau)^{-\gamma} + \beta \tau^{-\gamma} = 0\,.$$

To solve for τ we proceed as follows:

$$\begin{aligned}
(y_1 - \tau)^{-\gamma} &= \beta \tau^{-\gamma} \\
y_1 - \tau &= \beta^{-\frac{1}{\gamma}} \tau \\
y_1 &= \left[1 + \beta^{-\frac{1}{\gamma}}\right] \tau \\
\implies \quad \tau &= \frac{y_1}{1 + \beta^{-\frac{1}{\gamma}}}
\end{aligned} \tag{5.13}$$

What can we learn from the expression in (5.13)? First, consider the situation when $\beta = 1$. The optimal payroll tax then simplifies to

$$\tau = \frac{1}{2} y_1$$

A welfare-maximising government therefore taxes away half of the endowment when young and transfers it to the old in the same period. This is exactly what the utility-maximising household would do when $\beta = 1$.

Next, consider the case of log preferences with $\gamma = 1$. The payroll tax is then given by

$$\tau = \frac{1}{1+\beta^{-1}} y_1 = \frac{\beta}{1+\beta} y_1$$

and therefore the household consumes

$$c_1 = y_1 - \tau = y_1 - \frac{\beta}{1+\beta} y_1 = \frac{1}{1+\beta} y_1$$

while young. This is exactly the optimal consumption level we found in earlier units with log preferences when lifetime income was given by y_1. The conclusion is thus that the government can impose a policy that replicates the households' preferred allocation even in the absence of inter-cohort trade! ■

5.3.3 Pension system with endogenous labour supply

In the previous section, we had a government impose a payroll tax to finance old-age consumption. However, the (implicit) labour supply was assumed to be exogenous. In a more realistic setting with endogenous labour supply, we would expect payroll taxes to have an effect on the household's willingness to work, which in a production economy would also affect output.

As before, we impose that the government runs a balanced budget. Assume that τ is a proportional payroll tax, the wage rate is given by w, the household chooses to supply $1 - \ell$ units of labour (and hence chooses to consume ℓ units of leisure), and pensions T are transferred to the old household. Then the government budget balance requires that

$$\underbrace{T}_{\text{Pensions}} = \underbrace{\tau w(1-\ell)}_{\text{Payroll taxes}} . \tag{5.14}$$

How can the government use this pension system to transfer resources between cohorts, and what is the labour supply response of young households? We investigate one such model in the next example.

Example 5.7 (PAYGO with endogenous labour and log preferences). Consider a household with log preferences that lives for two periods and endogenously supplies labour while young:

$$\max_{c_1, c_2, a_2} \quad \log(c_1) + \log(\ell) + \beta \log(c_2)$$

$$\text{s.t.} \quad c_1 + a_2 = (1-\tau) w (1-\ell) \tag{5.15}$$

$$c_2 = (1+r) a_2 + T \tag{5.16}$$

$$\ell \in [0,1]$$

In addition to goods, the household now chooses the amount of leisure ℓ it wants to consume. Moreover, it pays a proportional payroll tax on labour income $w(1-\ell)$ while young, where w is the wage rate, and receives a lump-sum pension payment T when old.

The lifetime budget constraint is given by

$$c_1 + \frac{c_2}{1+r} = (1-\tau)w(1-\ell) + \frac{T}{1+r} \tag{5.17}$$

and the Lagrangian is

$$\mathcal{L} = \log(c_1) + \log(\ell) + \beta \log(c_2) + \lambda \left[(1-\tau)w(1-\ell) + \frac{T}{1+r} - c_1 + \frac{c_2}{1+r} \right] .$$

The first-order conditions w.r.t. c_1, c_2 and ℓ are

$$\frac{\partial \mathcal{L}}{\partial c_1} = \frac{1}{c_1} - \lambda = 0 \tag{5.18}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta \frac{1}{c_2} - \frac{\lambda}{1+r} = 0 \tag{5.19}$$

$$\frac{\partial \mathcal{L}}{\partial \ell} = \frac{1}{\ell} - \lambda(1-\tau)w = 0 \tag{5.20}$$

As usual, we obtain the Euler equation by combining (5.18) and (5.19):

$$\frac{1}{c_1} = \beta(1+r)\frac{1}{c_2}$$

while (5.18) and (5.20) together give the intra-temporal condition that equates the MRS between c_1 and ℓ to the relative price,

$$\underbrace{\frac{1/\ell}{1/c_1}}_{MRS_{c_1,\ell}} = \underbrace{\frac{(1-\tau)w}{1}}_{\text{Relative price}} \tag{5.21}$$

where the price of the consumption good is normalised to one. We can rewrite the intra-temporal optimality condition (5.21) as

$$c_1 = \ell(1-\tau)w \, . \tag{5.22}$$

In this example, we assume away government debt, so households won't be able to save in equilibrium for the reasons discussed earlier. We can therefore impose $a_2 = 0$ in the young household's budget constraint (5.15) and use (5.22) to get an equation in a single unknown, ℓ:

$$\begin{aligned} c_1 &= (1-\tau)w(1-\ell) \\ \ell(1-\tau)w &= (1-\tau)w(1-\ell) \\ \ell &= (1-\ell) \end{aligned}$$

Solving for ℓ therefore yields

$$\ell = \frac{1}{2}\,. \tag{5.23}$$

Consumption while young is thus given by

$$c_1 = \ell(1-\tau)w = \frac{1}{2}(1-\tau)w$$

whereas in old age, from (5.16) we have

$$c_2 = T = \tau(1-\ell)w = \frac{1}{2}\tau w$$

where the second equality follows from the government budget balance (5.14). To find the equilibrium interest rate, we insert consumption into the Euler equation and simplify,

$$\begin{aligned} \frac{1}{\frac{1}{2}(1-\tau)w} &= \beta(1+r)\frac{1}{\frac{1}{2}\tau w} \\ \frac{1}{1-\tau} &= \beta(1+r)\frac{1}{\tau} \end{aligned}$$

Solving for r yields

$$r = \frac{1}{\beta}\frac{\tau}{1-\tau} - 1 \tag{5.24}$$

So far, we ignored that w is also determined in general equilibrium. However, if we assume a production function of the form $f(L) = A \cdot L$, then $w = A$ in equilibrium, and consequently the wage rate is de facto exogenous and given by the productivity parameter A. Moreover, as you see from (5.23) and (5.24), neither the optimal choice of ℓ nor the interest rate r depend on the wage rate, irrespective of the production function. ■

Example 5.8 (Optimal payroll tax with endogenous labour supply). In Example 5.7, the policy parameter τ was assumed fixed. Which τ should a welfare-maximising government set? We again assume that the government takes as given optimal household choices and solves

$$\max_{\tau \in [0,1]} \ \log(c_1^*) + \log(\ell^*) + \beta \log(c_2^*)$$

where

$$c_1^* = \frac{1}{2}(1-\tau)w \tag{5.25}$$

$$c_2^* = \frac{1}{2}\tau w \tag{5.26}$$

$$\ell^* = \frac{1}{2}$$

which are the solutions found in general equilibrium in Example 5.7. We assume that the production function takes labour as the only argument and is given by $f(L) = A \cdot L$ so that $w = A$ is constant equilibrium.

Plugging the optimal household choices into the government's maximisation problem yields

$$\max_{\tau \in [0,1]} \ \log\left(\frac{1}{2}(1-\tau)w\right) + \beta \log\left(\frac{1}{2}\right) + \log\left(\frac{1}{2}\tau w\right)$$

Getting rid of all the constant terms which have no effect on the optimum, the Lagrangian can be stated as

$$\mathcal{L} = \log(1-\tau) + \beta \log(\tau) + \lambda_1 + \lambda_2(1-\tau)$$

where λ_1 and λ_2 are the Lagrange multipliers for the constraints $\tau \geq 0$ and $\tau \leq 1$. As before, we ignore these since we know that the optimal τ will satisfy $0 < \tau < 1$. The first-order condition for τ is thus

$$\frac{\partial \mathcal{L}}{\partial \tau} = -\frac{1}{1-\tau} + \beta \frac{1}{\tau} = 0 \tag{5.27}$$

Solving for τ, we find that the socially optimal payroll tax rate is

$$\begin{aligned}
\frac{1}{1-\tau} &= \beta \frac{1}{\tau} \\
1-\tau &= \frac{1}{\beta}\tau \\
1 &= \frac{1+\beta}{\beta}\tau \\
\implies \quad \tau &= \frac{\beta}{1+\beta}
\end{aligned}$$

We can get the implied equilibrium interest rate by substituting for τ in (5.24),

$$r^* = \frac{1}{\beta} \frac{\frac{\beta}{1+\beta}}{1 - \frac{\beta}{1+\beta}} - 1 = \frac{1}{\beta} \frac{\frac{\beta}{1+\beta}}{\frac{1}{1+\beta}} - 1 = \frac{1}{\beta} \frac{\beta}{1} - 1 = 0$$

Lastly, we get the welfare-maximising consumption allocation by plugging the expression for τ into (5.25) and (5.26)

$$\begin{aligned}
c_1 &= \frac{1}{2}(1-\tau)w = \frac{1}{1+\beta}\frac{1}{2}w \\
c_2 &= \frac{1}{2}\tau w = \frac{\beta}{1+\beta}\frac{1}{2}w
\end{aligned}$$

Note that since $r = 0$ at the optimum, the present value of lifetime income (both from labour income and pensions) is given by

$$\begin{aligned}(1-\tau)(1-\ell)w + \frac{T}{1+r} &= (1-\tau)(1-\ell)w + \frac{\tau(1-\ell)w}{1+r} \\ &= (1-\tau)(1-\ell)w + \tau(1-\ell)w \\ &= (1-\ell)w \\ &= \frac{1}{2}w\end{aligned}$$

where we have used the government budget constraint (5.14) and the optimal leisure choice $\ell = \frac{1}{2}$. Consequently, optimal consumption while young again coincided with what a household with log preferences would have chosen if it was possible to save.

■

5.4 Social planner solution

You might have noticed that in all three models with a welfare-maximising government, we arrived at the same equilibrium interest rate, $r^* = 0$. Moreover, the consumption allocation was identical (most easily seen for log preferences), even in the case of endogenous labour supply if we impose that labour productivity is given by $A = 2y_1$. Under these assumptions, in all three models we found that

$$\begin{aligned}c_1 &= \frac{1}{1+\beta}y_1 \\ c_2 &= \frac{\beta}{1+\beta}y_1\end{aligned}$$

in the case of log preferences. This is not a coincidence but arises because the government can arbitrarily shift consumption between young and old using its policy instrument (either b or τ), and the welfare-maximising allocation coincides with that of the social planner in each case, as we demonstrate in the next example.

Example 5.9 (Social planner solution). Assume that the social planner solves

$$\max_{c_1, c_2} \quad \log(c_1) + \beta \log(c_2) \tag{5.28}$$

$$\text{s.t.} \quad c_1 + c_2 = y_1 \tag{5.29}$$

This problem is conceptually different from the government's objective, because the social planner maximises welfare *at a point in time* by efficiently distributing the aggregate endowment y_1 between the old and the young. Conversely, the government maximised the welfare of one single cohort by solving a dynamic problem. However, these problems coincide if we assume that the social planner attaches a weight of β to old households whereas the weight of young households is one, as shown in (5.28).

The social planner's first-order conditions are

$$\frac{1}{c_1} = \lambda$$
$$\beta \frac{1}{c_2} = \lambda$$

which can be combined to eliminate λ:

$$\frac{1}{c_1} = \beta \frac{1}{c_2} \tag{5.30}$$

Substituting for $c_2 = \beta c_1$ in the aggregate resource constraint (5.29),

$$c_1 + \beta c_1 = y_1$$

we find that the young consume

$$c_1 = \frac{1}{1+\beta} y_1 ,$$

while consumption of the old is given by

$$c_2 = \frac{\beta}{1+\beta} y_1 .$$

This is exactly the consumption allocation we found in all three examples with a welfare-maximising government. What's more, if we compare (5.30) with the household's Euler equation, we see that these optimality conditions are identical if $r^* = 0$ in equilibrium. To put it differently, with $r^* = 0$, the households and the social planner's optimality conditions are aligned, and hence the government is able to bring about the first-best allocation. However, note that this result is specific to the simple models we studied here and does not necessarily generalise to more complex settings.

■

5.5 Main takeaways

We studied equilibria in OLG economies with two cohorts and concluded that

1. In pure endowment economies without a government, households are unable to smooth consumption over time because there is no trade in equilibrium.
2. Introducing a government allows us to study equilibria other than autarky.
3. A government can use several instruments to transfer resources across cohorts, for example by introducing government debt or a redistributive pension system.
4. By setting an optimal debt level or an optimal tax rate, the government achieves the first-best allocation which coincides with the planner's solution.

With more than two overlapping cohorts, it is possible to find equilibria with trade where some cohorts choose to borrow while others lend in order to save for retirement.

Appendix 5.1: Government debt with CRRA preferences

In this section, we generalise the model with government debt which we solved for log preferences in section 5.3.1 to the CRRA case with $\gamma \neq 1$. Additionally, we allow for a non-zero endowment in old age.

Household problem (partial equilibrium). The household now solves

$$\max_{c_1, c_2, a_2} \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \frac{c_2^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad c_1 + a_2 = y_1 - \tau \tag{5.31}$$

$$c_2 = (1+r)a_2 + y_2 \tag{5.32}$$

Combining the budget constraints (5.31) and (5.32), the lifetime budget constraint reads

$$c_1 + \frac{c_2}{1+r} = y_1 - \tau + \frac{y_2}{1+r} \,. \tag{5.33}$$

The Euler equation for the CRRA case is standard,

$$c_1^{-\gamma} = \beta(1+r)c_2^{-\gamma} \tag{5.34}$$

which can be solved for c_2,

$$c_2 = \big[\beta(1+r)\big]^{\frac{1}{\gamma}} c_1 \,.$$

Substituting for c_2 in the lifetime budget constraint (5.33) gives

$$c_1 + \frac{\big[\beta(1+r)\big]^{\frac{1}{\gamma}} c_1}{1+r} = y_1 - \tau + \frac{y_2}{1+r} \,.$$

Solving for c_1, we find that

$$c_1 + \beta^{\frac{1}{\gamma}}(1+r)^{\frac{1}{\gamma}-1} c_1 = y_1 - \tau + \frac{y_2}{1+r}$$

$$c_1 = \frac{1}{1+\beta^{\frac{1}{\gamma}}(1+r)^{\frac{1-\gamma}{\gamma}}} \left[y_1 - \tau + \frac{y_2}{1+r}\right] \tag{5.35}$$

General equilibrium. We use the household's Euler equation to pin down the equilibrium interest rate. Using the bond market clearing, $a_2 = b$, and the government budget balance, $rb = \tau$, we can write consumption while young as

$$c_1 = y_1 - \tau - a_2 = y_1 - rb - b = y_1 - (1+r)b \,.$$

Substituting for c_1 and c_2 in (5.34), we have

$$\Big[y_1 - (1+r)b\Big]^{-\gamma} = \beta(1+r)\Big[(1+r)b + y_2\Big]^{-\gamma}$$

We can simplify this expression somewhat to read

$$y_1 - (1+r)b = \left[\beta(1+r)\right]^{-\frac{1}{\gamma}}\left[(1+r)b + y_2\right]$$

This still leaves us with a non-linear equation in r which can only be solved numerically. Letting $y_2 = 0$ and $\gamma = 1$ as in the main text, we see that the expression simplifies to

$$\frac{y_1}{b} - (1+r) = \frac{1}{\beta} \quad \Longrightarrow \quad r = \frac{y_1}{b} - \frac{1+\beta}{\beta}$$

which is exactly what we found in (5.9). Once we have found r numerically, we can recover the tax from the government budget balance $\tau = r \cdot b$ and plug these quantities into (5.35) to get c_1. Old-age consumption c_2 follows from the budget constraint (5.32).

To illustrate the difference to the log case, Figure 5.9 plots the allocation of (c_1, c_2) against the debt-to-income ratio for an RRA of $\gamma = 2$. Recall that with a higher RRA, the elasticity of intertemporal substitution $EIS = \frac{1}{\gamma}$ is lower and hence the household smooths consumption across periods more than in the log case. This can be seen by comparing the consumption allocation from Figure 5.9 to Figure 5.5. Finally, Figure 5.10 shows the equilibrium income tax and interest rate for the economy with $\gamma = 2$.

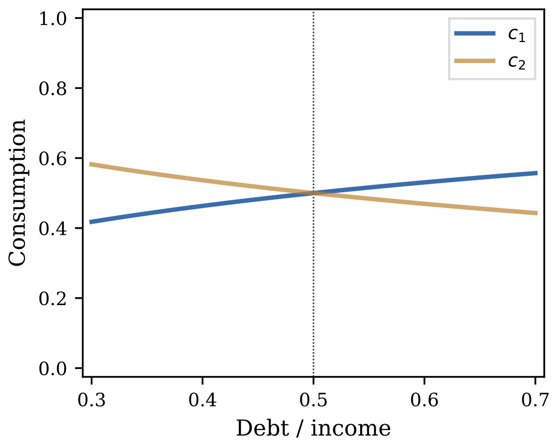
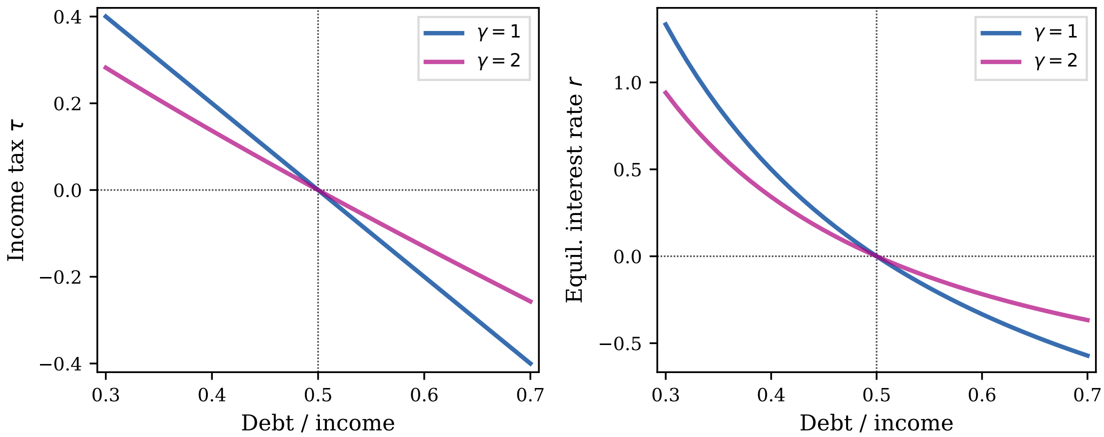


Figure 5.8: Optimal consumption

Figure 5.9: Optimal consumption plotted against the debt-to-income ratio b/y_1 for $\beta = 1$, $\gamma = 2$, $y_1 = 1$ and $y_2 = 0$.



(a) Income tax τ **(b)** Interest rate r

Figure 5.10: Tax level and equilibrium interest rate plotted against the debt-to-income ratio b/y_1 for $\beta = 1$, $y_1 = 1$ and $y_2 = 0$.