Results from the Dixit/Stiglitz monopolistic competition model

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Contents

1 Introduction

The Dixit/Stiglitz monopolistic competition model has been widely adopted in various fields of economic research such as international trade. The Dixit and Stiglitz (1977) paper actually contains three distinct models, yet economic literature has mostly taken up only the first one (constant elasticity case) and its market equilibrium solution. The main results of this subset of Dixit and Stiglitz (1977) are derived and explained below in order to aid in understanding this widespread model.

2 Constant elasticity sub-utility function

2.1 Preferences and demand

Assumption 2.1. Preferences are given by a (weakly-)separable, convex utility function

$$u = U(x_0, V(x_1, x_2, \dots, x_n))$$

where $U(\cdot)$ is either a social indifference curve or the multiple of a representative consumer's utility. x_0 is a numeraire good produced in one sector while $x_1, x_2, \dots, x_n$ are differentiated goods produced in another sector.[1]

[1] With weakly-separable utility functions, the MRS (and thus the elasticity of substitution) of two goods from the same group is independent of the quantities of goods in other sub-groups (see Gravelle and Rees 2004, 67).

Dixit and Stiglitz (1977) treat three different cases in which they alternately impose two of the three following restrictions:

1. symmetry of $V(\cdot)$ w.r.t. to its arguments;
2. CES specification for $V(\cdot)$
3. Cobb-Douglas form for $U(\cdot)$

However, throughout the literature many authors have imposed all three restrictions together in what Neary (2004) calls “Dixit-Stiglitz lite”.

The next section examines the more general case in which only restrictions 1 and 2 are imposed. Subsection 2.1.2 briefly looks and the more special case of “Dixit-Stiglitz lite”.

2.1.1 General case

First-stage optimization.

Assumption 2.2. In the CES case the utility function is given as

$$u = U\left(x_0, \left[\sum_i x_i^\rho\right]^{1/\rho}\right)$$

with $\rho \in (0,1)$ to allow for zero quantities and ensure concavity of $U(\cdot)$. ρ is called the substitution or “love-of-variety” parameter.[2] Furthermore, $U(\cdot)$ is assumed to be *homothetic* in its arguments.

Since $U(\cdot)$ is a separable utility function, the consumer optimization problem can be solved in two separate steps: first the optimal allocation of income for each subgroup is determined, then the quantities within each subgroup.

Definition 2.1. Let y be a quantity index presenting all goods $x_1, x_2, \ldots, x_n$ from the second sector such that

$$y \equiv \left[\sum_i x_i^\rho\right]^{1/\rho} . \tag{1}$$

Figure 1 shows some illustrative examples for the two-dimensional case. As required, the utility function is concave and x_1, x_2 are neither complements nor perfect substitutes if $\rho \in (0,1)$.

The first-stage optimization problem is given as follows:

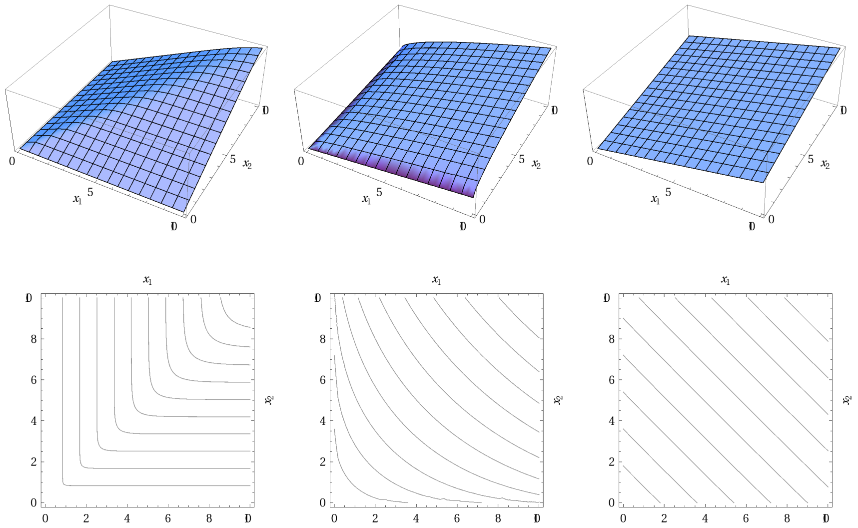
$$\begin{aligned} \max. \quad & U(x_0, y) \\ \text{s.t.} \quad & x_0 + q \cdot y = I \end{aligned}$$

where I is the income in terms of the numeraire and q is the price index of y.[3]

[2] With $\rho = 1$, $x_1, \ldots, x_n$ are perfect substitutes as the subutility function simplifies to $V(x_1, x_2, \ldots, x_n) = \sum_i x_i$ and thus it does not matter which x_i is consumed. For $\rho < 0$ they are complements (see Brakman et al. 2001, 68).

[3] The income consists of an initial endowment which is normalized at 1, plus firm profits distributed to consumers or minus a lump sum required to cover firm losses. However, since the discussion here is limited to the market equilibrium case, firms make zero profit so $I = 1$.

Figure 1: *CES functions with two-dimensional domain and varying ρ parameter (from left to right: $\rho = \{$-10, 0.5, 0.99$\}$). The first case is excluded in this model due to the restrictions on parameter ρ.*



From the Lagrangian $\mathcal{L} = U(x_0, y) - \lambda[x_0 - qy - I]$ we obtain the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_0} = U_0 - \lambda = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial y} = U_y - \lambda q = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_0 + qy - I = 0$$

From (2) and (3) we get the necessary condition

$$\frac{U_y}{U_0} = q = \frac{p_y}{p_0} \tag{4}$$

which is the familiar $U_i/U_j = p_i/p_j$ as q is the price index for y and $p_0 = 1$ due to the numeraire definition. Since $U(\cdot)$ was assumed homothetic, this uniquely identifies the share of expenditure for x_0 and y because these solely depend on relative marginal utilities. Denote the share of expenditure on y as $s(q)$ and that on x_0 as $(1-s(q))$. Then the optimal quantities for each sector are

$$\begin{aligned} x_0 &= (1 - s(q))I \\ y &= \frac{s(q)I}{q} \end{aligned} \tag{5}$$

Second-stage optimization. Given the definition of y and $s(q)$, the second-state problem is

$$\begin{aligned} \text{max.} \quad & y = \left[\sum_i x_i^{\rho}\right]^{1/\rho} \\ \text{s.t.} \quad & \sum_i p_i x_i = s(q)I \end{aligned}$$

The Lagrangian $\mathcal{L} = (\sum_i x_i^\rho)^{1/\rho} - \lambda(\sum_i p_i x_i - s(q)I)$ yields the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_i} = y^{1-\rho} x_i^{\rho-1} - \lambda p_i = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_i p_i x_i - s(q)I = 0 \tag{7}$$

Solving (6) for x_i gives

$$x_i = y(\lambda p_i)^{1/(\rho-1)} \tag{8}$$

Inserting this into (7) and solving for λ we get

$$\sum_i p_i y(\lambda p_i)^{1/(\rho-1)} = s(q)I$$

$$\lambda^{1/(\rho-1)} y \sum_i p_i^{\rho/(\rho-1)} = s(q)I$$

$$\lambda^{1/(\rho-1)} = \frac{s(q)I)}{y} \left[\sum_i p_i^{\rho/(\rho-1)} \right]^{-1} \tag{9}$$

Finally, plugging (9) back into (8) we get the preliminary demand function facing a single firm in the second sector:[4]

$$x_i = s(q)I p_i^{1/(\rho-1)} \left[\sum_j p_j^{\rho/(\rho-1)} \right]^{-1} \tag{10}$$

To further simplify this expression, take (10) to the power of ρ and sum over i:[5]

$$x_i^\rho = (s(q)I)^\rho p_i^{\rho/(\rho-1)} \left[\sum_j p_j^{\rho/(\rho-1)} \right]^{-\rho}$$

$$\sum_i x_i^\rho = \left[\sum_j p_j^{\rho/(\rho-1)} \right]^{-\rho} (s(q)I)^\rho \sum_i p_i^{\rho/(\rho-1)}$$

$$y = \left[\sum_i x_i^\rho \right]^{1/\rho} = \left[\sum_j p_j^{\rho/(\rho-1)} \right]^{-1} s(q)I \left[\sum_i p_i^{\rho/(\rho-1)} \right]^{1/\rho}$$

$$y = \frac{s(q)I}{\left[\sum_i p_i^{\rho/(\rho-1)} \right]^{(\rho-1)/\rho}} \tag{11}$$

From (5) and (11) we obtain

Result 2.1. For the utility function given in Assumption 2.2 and the composite quantity index from Definition 2.1 the corresponding *price index* is

$$q = \left[\sum_i p_i^{\rho/(\rho-1)} \right]^{(\rho-1)/\rho} \tag{12}$$

[4] The summation index has been changed to j to reflect that it in is unrelated to i.

[5] The last step follows from the fact that both sums are identical, regardless whether the index i or j is used.

Using Result 2.1, (10) can be simplified to arrive at

Result 2.2. For the utility function given in Assumption 2.2, the resulting *demand function* facing a single firm is

$$x_i = y \left[\frac{q}{p_i} \right]^{1/(1-\rho)} \tag{13}$$

with y and q defined in (1) and (12), respectively.

Some remarks regarding the demand function and CES preferences are in order.

First, by plugging $y = s(q)I/q$ into (13) and taking logs, it can easily be seen that the varieties $x_1, x_2, \dots, x_n$ have unit *income elasticities* $\partial \log x_i / \partial \log I$.

Second, assuming a sufficiently large number of varieties so that pricing decisions of a single firm do not affect the general price index, the *price elasticity of demand* for x_i is

$$\epsilon_d = \left. \frac{\partial \log x_i}{\partial \log p_i} \right|_{q \text{ const.}} = \frac{1}{\rho - 1} \tag{14}$$

At this point it is convenient to define $\sigma \equiv 1/(1-\rho)$ so that $\epsilon_d = -\sigma$.[6]

Third, to get the *elasticity of substitution* between two varieties, from (6) we see that

$$\frac{x_i}{x_j} = \left[\frac{p_j}{p_i} \right]^{1/(1-\rho)}$$

and hence the elasticity of substitution can be obtained as

$$\epsilon_s = \frac{\partial \log(x_j/x_i)}{\partial \log(p_i/p_j)} = \frac{\partial \log(p_i/p_j)^{1/(1-\rho)}}{\partial \log(p_i/p_j)} = \frac{1}{1-\rho} = \sigma \tag{15}$$

This can be summarized as

Result 2.3. Dixit-Stiglitz preferences given in Assumption 2.2 result in *constant* demand and substitution elasticities given by

$$\epsilon_d = \frac{1}{\rho - 1} = -\sigma \qquad \epsilon_s = \frac{1}{1-\rho} = \sigma$$

Often the model is specified directly in terms of σ instead of ρ,[7] with

$$u = U(x_0, y) \ , \qquad y \equiv \left[\sum_i x_i^{1-1/\sigma} \right]^{1/(1-1/\sigma)} \quad , \qquad q \equiv \left[\sum_i p_i^{1-\sigma} \right]^{1/(1-\sigma)}$$

Fourth, to see why the CES utility specification is called "variety-loving", inspect the large-subgroup case with many varieties n with similar price levels, i.e. $p_i \approx p$ and hence $x_i = x$. Then expenditure is equally divided over all varieties $x_1, x_2, \dots, x_n$ since they symmetrically

[6] Here σ is different from $\sigma(q)$ in Dixit and Stiglitz (1977), but reflects the notation of many other Dixit-Stiglitz-based models.

[7] For example, see Baldwin et al. (2005, 38).

enter into the subutility function. If there exist n varieties, the expressions for y and q simplify to

$$y = \left[\sum_i^n x^\rho\right]^{1/\rho} = xn^{1/\rho} \tag{16}$$

$$q = \left[\sum_i^n p^{\rho/(\rho-1)}\right]^{(\rho-1)/\rho} = pn^{(\rho-1)/\rho} \tag{17}$$

Plugging (5) and (17) into (13) gives a simplified demand function for the large-subgroup case:

$$x = \frac{s(q)I}{np} \ . \tag{18}$$

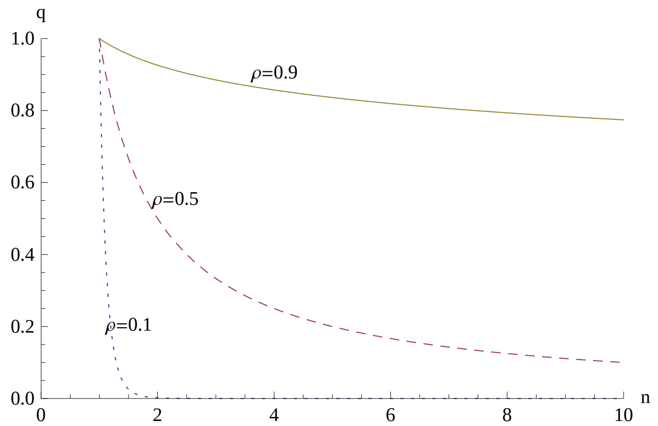
Substituting this for x in the subutility function (1), we obtain

$$\begin{aligned} V(n) = y &= \left[\sum_i^n \left[\frac{s(q)I}{np}\right]^\rho\right]^{1/\rho} \\ &= n^{(1/\rho)}\frac{s(q)I}{np} \\ &= n^{1/\rho-1}(nx) \end{aligned}$$

which is increasing in n as $\rho \in (0,1)$ by assumption. The last equality provides some intuitive insights: since (nx) is the actual quantity produced, the term $n^{1/\rho-1} > 1$ can be seen as an additional "bonus", so variety represents an externality or the extent of the market. Increasing the market size nx has a more than proportional effect on utility due to this term (Brakman et al. 2001, 68).

That utility increases with variety can also be seen by recalling that $V(x) = y = s(q)I/q$ and examining how q from (17) changes with n, as shown in Figure 2.

Figure 2: *Price index q as a function of the number of varieties n (assuming $p = 1$)*



It is evident that for constant expenditure, the price index falls rapidly, with utility rising as a consequence. This effect is more pronounced for ρ closer to 0, which can intuitively be explained using Figure 1: for ρ close to 1, all varieties are close substitutes and hence introducing another similar variety only moderately increases utility. The converse is true for ρ close to 0.

2.1.2 Cobb-Douglas case ("Dixit-Stiglitz lite")

In this section we inspect a special case of the model in which all three of the initially mentioned restrictions on utility are imposed.

Assumption 2.3. If $U(\cdot)$ is Cobb-Douglas and $V(\cdot)$ is CES, the resulting utility function is given by

$$u = U(x_0, y) = x_0^{1-\alpha} y^{\alpha}.$$

Again a two-state optimization approach is applicable.

First-stage optimization. The maximization problem is stated as follows:

$$\begin{aligned} \max. \quad & u = x_0^{1-\alpha} y^{\alpha} \\ \text{s.t.} \quad & x_0 + qy = I \end{aligned} \tag{19}$$

As in (4), the necessary condition from the Lagrangian is $U_y/U_0 = q$, which together with the budget constraint yields the well-known result for Cobb-Douglas utility:

$$x_0 = (1-\alpha)I \tag{20}$$

$$y = \frac{\alpha I}{q} \tag{21}$$

Second-stage optimization. From here the second-stage optimization proceeds exactly as in the general case, with α replacing $s(q)$. Using the definition of q one arrives at the demand function given in Result 2.2.

With the Cobb-Douglas / CES case it can easily be verified that a single-stage optimization process yields the same results. The Lagrangian in this case is

$$\mathcal{L} = x_0^{1-\alpha} \left[\sum_i x_i^{\rho} \right]^{\alpha/\rho} - \lambda \left[x_0 + \sum_i p_i x_i - I \right]$$

with the relevant first-order condition being

$$\frac{\partial \mathcal{L}}{\partial x_i} = x_0^{(1-\alpha)} \frac{\alpha}{\rho} \left[\sum_i x_i^{\rho} \right]^{(\alpha-\rho)/\rho} \rho x_i^{\rho-1} - \lambda p_i = 0$$

Dividing the first-order conditions for x_i and x_j, multiplying by p_i and summing over i the demand function can be obtained:

$$\begin{aligned}
\frac{x_i}{x_j} &= \left[\frac{p_i}{p_j}\right]^{1/(\rho-1)} \\
p_i x_i &= p_i^{\rho/(\rho-1)} p_j^{1/(1-\rho)} x_j \\
I - x_0 = \sum_i x_i p_i &= p_j^{1/(1-\rho)} x_j \sum_i p_i^{\rho/(\rho-1)} \\
x_j &= \frac{(I - x_0) p_j^{1/(\rho-1)}}{\sum_i p_i^{\rho/(\rho-1)}} \\
&= \frac{I - x_0}{q} \frac{q^{1/(1-\rho)}}{p_j^{1/(1-\rho)}} \qquad \text{[by def. of } q\text{]} \\
&= y \left[\frac{q}{p_j}\right]^{1/(1-\rho)}
\end{aligned}$$

2.2 Firms and production

It is assumed that all firms producing varieties of x_i have identical fixed and marginal costs. Since consumers demand all existing varieties symmetrically, any new firm entering the market will choose to produce a unique variety and exploit monopolistic pricing power instead of entering into a duopoly with an existing producer. Also, every firm will choose to produce one variety only (see Baldwin et al. (2005, 42) on how to derive this result).

Production for each firm exhibits (internal) increasing returns to scale. This is implied by introducing fixed costs in addition to (constant) marginal costs as stated above. Hence the cost function has the form

$$C(x) = cx + F \tag{22}$$

where c is the marginal costs and F the fixed cost per variety (there are no economies of scope).[8]

2.3 Market equilibrium

Equilibrium in this model is determined by two conditions: first, firms maximize profits consistent with the demand function (13); second, as this creates pure profit which induces new firms to enter the market, quantities of x_i adjust until the marginal firm just breaks even (free entry condition).

Profit maximization. Since each firm produces a unique variety, monopolistic pricing applies and each firm faces the maximization problem

$$\max. \qquad \pi = p(x)x - cx - F \tag{23}$$

It is assumed that each firm takes price setting behavior of other firms as given (other firms do not adapt their prices as a reaction to the firm's price) and that firms ignore effects of their pricing decisions on the price index q. Again, this assumption is only plausible with a sufficiently large number of firms.

[8] As all firms have identical cost functions, face identical demand functions and all varieties enter symmetrically into the utility function, subscripts i will be omitted from now on, i.e. $x_i = x$, $p_i = p$.

The necessary first-order condition resulting from (23) is the well-known

$$p\left[1+\frac{1}{\epsilon_d}\right]=c$$
$$p\left[1-\frac{1}{\sigma}\right]=c$$

where ϵ_d is the elasticity of demand, which was shown to equal $-\sigma = 1/(\rho-1)$ in Result 2.3. Solving for p, we obtain

Result 2.4. In equilibrium, the optimal price is given by

$$p_e = \frac{c}{\rho}\ ,$$

where p_e is calculated as a *constant* mark-up over marginal cost c.

Free entry condition. As the model assumes free entry, new firms will enter the market and produce a new variety as long as this yields positive profit. When a firm enters the market and starts producing a new variety, consumers divert some of the expenditure previously spent on existing varieties to purchase the new good. The quantity of each variety sold decreases, as does profit due to rising average costs. As a consequence, the free entry condition states that in equilibrium the marginal firm (indexed by n) just breaks even, i.e. operating profit equals fixed cost:[9]

$$(p_n - c)x_n = F \tag{24}$$

With symmetry and identical firms, condition (24) holds for all intramarginal firms as well. Solving (24) for x, we get

Result 2.5. The free entry condition dictates that in equilibrium the quantity of each variety produced is

$$x_e = \frac{F}{p_e - c} = \frac{F}{c}(\sigma - 1)\ . \tag{25}$$

Naturally, in equilibrium the number of varieties produced has to be consistent with the demand function from (18), and therefore

$$\frac{s(p_e n_e^{(\rho-1)/\rho})}{p_e n_e} = \frac{F}{(p_e - c)} \tag{26}$$

must hold. This uniquely identifies an equilibrium if the left-hand side is a monotonic function of n, which is the case if the elasticity w.r.t. n has a determinate sign. It is assumed to be negative as the quantity of each variety consumed decreases when more varieties are available. See Dixit and Stiglitz (1977, 300) for a formal condition for this to hold.

Before finishing this section, some further remarks regarding the equilibrium are necessary. First, from (25) it can be seen that equilibrium quantities are constant and depend on the two cost parameters, F and c, and on one demand parameter, σ, all of which are exogenously determined. They are independent of other factors such as the number of varieties produced. Therefore, aggregate manufacturing output can only increase by increasing the number of

[9] Ignoring integer constraints, the number of firms n is assumed to be large enough that this can be stated as an equality.

varieties. This determines the outcome of models such as Krugman (1980), where increasing the market size via trade liberalization results in more varieties, not higher quantities per firm.[10]

Second, calculating equilibrium operating profit (ignoring fixed costs) as

$$\pi_e = (p_e - p_e\rho)x_e$$
$$\pi_e = (1-\rho)p_e x_e$$
$$\pi_e = \frac{p_e x_e}{\sigma}$$

we see that operating profits are determined as a constant profit margin $1/\sigma$ of revenue $p_e x_e$.

[10] However, this result depends on several assumptions, viz. ice-berg trade costs, homothetic cost functions and mill pricing, i.e. invariant mark-ups (see Baldwin et al. 2005, 42).

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